CHAPTER

2

LOGIC

Mathematicians throughout the centuries have used logic as the foundation of their understanding of the relationships among mathematical ideas. In the late 17th century, Gottfried Leibniz (1646–1716) organized logical discussion into a systematic form, but he was ahead of the mathematical thinking of his time and the value of his work on logic was not recognized.

It was not until the 19th century that George Boole (1815–1864), the son of an English shopkeeper, developed logic in a mathematical context, representing sentences with symbols and carefully organizing the possible relationships among those sentences.

Boole corresponded with Augustus DeMorgan (1806–1871) with whom he shared his work on logic. Two important relationships of logic are known today as DeMorgan's Laws.

$$\text{not } (p \text{ and } q) = (\text{not } p) \text{ or } (\text{not } q)$$

$$\text{not } (p \text{ or } q) = (\text{not } p) \text{ and } (\text{not } q)$$

Boolean algebra is key in the development of computer science and circuit design.

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2-1 SENTENCES, STATEMENTS, AND TRUTH VALUES

Logic is the science of reasoning. The principles of logic allow us to determine if a statement is true, false, or uncertain on the basis of the truth of related statements.

We solve problems and draw conclusions by reasoning from what we know to be true. All reasoning, whether in mathematics or in everyday living, is based on the ways in which we put sentences together.

Sentences and Their Truth Values

When we can determine that a statement is true or that it is false, that statement is said to have a **truth value**. In this chapter we will study the ways in which statements with known truth values can be combined by the *laws of logic* to determine the truth value of other statements.

In the study of logic, we use simple declarative statements that state a fact. That fact may be either true or false. We call these statements **mathematical sentences**. For example:

1. Congruent angles are angles that have the same measure.	True mathematical sentence
2. $17 - 5 = 12$	True mathematical sentence
3. The Brooklyn Bridge is in New York.	True mathematical sentence
4. $17 + 3 = 42$	False mathematical sentence
5. The Brooklyn Bridge is in California.	False mathematical sentence

Nonmathematical Sentences and Phrases

Sentences that do not state a fact, such as questions, commands, or exclamations, are not sentences that we use in the study of logic. For example:

1. Did you have soccer practice today?
This is not a mathematical sentence because it asks a question.

2. Go to your room.
This is not a mathematical sentence because it gives a command.

An incomplete sentence or part of a sentence, called a **phrase**, is not a mathematical sentence and usually has no truth value. For example:

1. Every parallelogram
This is not a mathematical sentence.

2. $19 - 2$
This is not a mathematical sentence.

Some sentences are true for some persons and false for others. For example:

1. I enjoy reading historical novels.
2. Summer is the most pleasant season.
3. Basketball is my favorite sport.

Conclusions based on sentences such as these do not have the same truth value for all persons. We will not use sentences such as these in the study of logic.

Open Sentences

In the study of algebra, we worked with **open sentences**, that is, sentences that contain a variable. The truth value of the open sentence depended on the value of the variable. For example, the open sentence $x + 2 = 5$ is true when $x = 3$ and false for all other values of x. In some sentences, a pronoun, such as *he*, *she*, or *it*, acts like a variable and the name that replaces the pronoun determines the truth value of the sentence.

1. $x \div 2 = 8$ — Open sentence: the variable is x.
2. He broke my piggybank. — Open sentence: the variable is *he*.
3. Jenny found it behind the sofa. — Open sentence: the variable is *it*.

In previous courses, we learned that the **domain** or **replacement set** is the set of all elements that are possible replacements for the variable. The element or elements from the domain that make the open sentence true is the **solution set** or **truth set**. For instance:

Open sentence: $14 - x = 9$
Variable: x
Domain: $\{1, 2, 3, 4, 5\}$
Solution set: $\{5\}$
When $x = 5$, then $14 - 5 = 9$ is a true sentence.

The method we use for sentences in mathematics is the same method we apply to sentences in ordinary conversation. Of course, we would not use a domain like $\{1, 2, 3, 4\}$ for the open sentence "It is the third month of the year." Common sense tells us to use a domain consisting of the names of months. The following example compares this open sentence with the algebraic sentence used above. Open sentences, variables, domains, and solution sets behave in exactly the same way.

Open sentence: It is the third month of the year.
Variable: It
Domain: {Names of months}
Solution set: {March}
When "It" is replaced by "March," then "March is the third month of the year" is a true sentence.

Sometimes a solution set contains more than one element. If Elaine has two brothers, Ken and Kurt, then for her, the sentence "He is my brother" has the solution set {Ken, Kurt}. Here the domain is the set of boys' names. Some people have no brothers. For them, the solution set for the open sentence "He is my brother" is the empty set, $\varnothing$ or { }.

EXAMPLE 1

Identify each of the following as a true sentence, a false sentence, an open sentence, or not a mathematical sentence at all.

	Answers
a. Football is a water sport.	False sentence
b. Football is a team sport.	True sentence
c. He is a football player.	Open sentence: the variable is *he*.
d. Do you like football?	Not a mathematical sentence: this is a question.
e. Read this book.	Not a mathematical sentence: this is a command.
f. $3x - 7 = 11$	Open sentence: the variable is x.
g. $3x - 7$	Not a mathematical sentence: this is a phrase or a binomial.

EXAMPLE 2

Use the replacement set $\left\{2, \frac{2\pi}{3}, 2.5, 2\sqrt{2}\right\}$ to find the truth set of the open sentence "It is an irrational number."

Solution Both 2 and 2.5 are rational numbers.

Both π and $\sqrt{2}$ are irrational numbers. Since the product or quotient of a rational number and an irrational number is an irrational number, $\frac{2\pi}{3}$ and $2\sqrt{2}$ are irrational.

Answer $\left\{\frac{2\pi}{3}, 2\sqrt{2}\right\}$

Statements and Symbols

A sentence that can be judged to be true or false is called a **statement** or a **closed sentence**. In a statement, there are no variables.

A closed sentence is said to have a *truth value*. The truth values, true and false, are indicated by the symbols **T** and **F**.

Negations

In the study of logic, you will learn how to make new statements based upon statements that you already know. One of the simplest forms of this type of reasoning is negating a statement.

The **negation** of a statement always has the opposite truth value of the given or original statement and is usually formed by adding the word *not* to the given statement. For example:

1. Statement: Neil Armstrong walked on the moon. (True)
Negation: Neil Armstrong did *not* walk on the moon. (False)

2. Statement: A duck is a mammal. (False)
Negation: A duck is *not* a mammal. (True)

There are other ways to insert the word *not* into a statement to form its negation. One method starts the negation with the phrase "It is not true that . . ." For example:

3. Statement: A carpenter works with wood. (True)
Negation: It is *not* true that a carpenter works with wood. (False)
Negation: A carpenter does *not* work with wood. (False)

Both negations express the same false statement.

Logic Symbols

The basic element of logic is a simple declarative sentence. We represent this basic element by a single, lowercase letter. Although any letter can be used to represent a sentence, the use of p, q, r, and s are the most common. For example, p might represent "Neil Armstrong walked on the moon."

The symbol that is used to represent the negation of a statement is the symbol $\sim$ placed before the letter that represents the given statement. Thus, if p represents "Neil Armstrong walked on the moon," then $\sim p$ represents "Neil Armstrong did *not* walk on the moon." The symbol $\sim p$ is read "not p."

Symbol	*Statement in Words*	*Truth Value*
p	There are 7 days in a week.	True
$\sim p$	There are not 7 days in a week.	False
q	$8 + 9 = 16$	False
$\sim q$	$8 + 9 \neq 16$	True

When p is true, then its negation $\sim p$ is false. When q is false, then its negation $\sim q$ is true.

► **A statement and its negation have opposite truth values.**

It is possible to use more than one negation in a sentence. Each time another negation is included, the truth value of the statement will change. For example:

Symbol	*Statement in Words*	*Truth Value*
r	A dime is worth 10 cents.	True
$\sim r$	A dime is not worth 10 cents.	False
$\sim(\sim r)$	It is not true that a dime is not worth 10 cents.	True

We do not usually use sentences like the third. Note that just as in the set of real numbers, $-(-a) = a$, $\sim(\sim r)$ always has the same truth value as r. We can use r in place of $\sim(\sim r)$. Therefore, we can negate a sentence that contains the word *not* by omitting that word.

$\sim r$: A dime is not worth 10 cents.
$\sim(\sim r)$: A dime is worth 10 cents.

The relationship between a statement p and its negation $\sim p$ can be summarized in the table at the right. When p is true, $\sim p$ is false. When p is false, $\sim p$ is true.

p	$\sim p$
T	F
F	T

EXAMPLE 3

In this example, symbols are used to represent statements. The truth value of each statement is given.

k: Oatmeal is a cereal. (True)
m: Massachusetts is a city. (False)

For each sentence given in symbolic form:

a. Write a complete sentence in words to show what the symbols represent.

b. Tell whether the statement is true or false.

	Answers	
(1) $\sim k$	**a.** Oatmeal is not a cereal.	**b.** False
(2) $\sim m$	**a.** Massachusetts is not a city.	**b.** True

Exercises

Writing About Mathematics

1. Explain the difference between the use of the term "sentence" in the study of grammar and in the study of logic.

2. a. Give an example of a statement that is true on some days and false on others.

b. Give an example of a statement that is true for some people and false for others.

c. Give an example of a statement that is true in some parts of the world and false in others.

Developing Skills

In 3–10, tell whether or not each of the following is a mathematical sentence.

3. Thanksgiving is on the fourth Thursday in November.

4. Albuquerque is a city in New Mexico.

5. Where did you go?

6. Twenty sit-ups, 4 times a week

7. Be quiet.

8. If Patrick leaps

9. $y - 7 = 3y + 4$

10. Tie your shoe.

In 11–18, all of the sentences are open sentences. Find the variable in each sentence.

11. She is tall.

12. We can vote at age 18.

13. $2y \geq 17$

14. $14x \div 8 = 9$

15. This country has the third largest population.

16. He hit a home run in the World Series.

17. It is my favorite food.

18. It is a fraction.

In 19–26: **a.** Tell whether each sentence is true, false, or open. **b.** If the sentence is an open sentence, identify the variable.

19. The Statue of Liberty was given to the United States by France.

20. They gained custody of the Panama Canal on December 31, 1999.

21. Tallahassee is a city in Montana.

22. A pentagon is a five-sided polygon.

23. $6x + 4 = 16$

24. $6(10) + 4 = 16$

25. $6(2) + 4 = 16$

26. $2^3 = 3^2$

In 27–31, find the truth set for each open sentence using the replacement set {Nevada, Illinois, Massachusetts, Alaska, New York}.

27. Its capital is Albany.

28. It does not border on or touch an ocean.

29. It is on the east coast of the United States.

30. It is one of the states of the United States of America.

31. It is one of the last two states admitted to the United States of America.

In 32–39, use the domain {square, triangle, rectangle, parallelogram, rhombus, trapezoid} to find the truth set for each open sentence.

32. It has three and only three sides.

33. It has exactly six sides.

34. It has fewer than four sides.

35. It contains only right angles.

36. It has four sides that are all equal in measure.

37. It has two pairs of opposite sides that are parallel.

38. It has exactly one pair of opposite sides that are parallel.

39. It has interior angles with measures whose sum is 360 degrees.

In 40–47, write the negation of each sentence.

40. The school has an auditorium.

41. A stop sign is painted red.

42. The measure of an obtuse angle is greater than 90°.

43. There are 1,760 yards in a mile.

44. Michigan is not a city.

45. $14 \times 2 - 16 = 12$

46. $3 + 4 + 5 \neq 6$

47. Today is not Wednesday.

In 48–56, for each given sentence: **a.** Write the sentence in symbolic form using the symbols shown below. **b.** Tell whether the sentence is true, false, or open.

Let p represent "A snake is a reptile."
Let q represent "A frog is a snake."
Let r represent "Her snake is green."

48. A snake is a reptile.

49. A snake is not a reptile.

50. A frog is a snake.

51. A frog is not a snake.

52. Her snake is green.

53. Her snake is not green.

54. It is not true that a frog is a snake.

55. It is not the case that a snake is not a reptile.

56. It is not the case that a frog is not a snake.

In 57–64, the symbols represent sentences.

p: Summer follows spring.
q: August is a summer month.
r: A year has 12 months.
s: She likes spring.

For each sentence given in symbolic form: **a.** Write a complete sentence in words to show what the symbols represent. **b.** Tell whether the sentence is true, false, or open.

57. $\sim p$

58. $\sim q$

59. $\sim r$

60. $\sim s$

61. $\sim(\sim p)$

62. $\sim(\sim q)$

63. $\sim(\sim r)$

64. $\sim(\sim s)$

2-2 CONJUNCTIONS

We have identified simple sentences that have a truth value. Often we wish to use a connecting word to form a compound sentence. In mathematics, sentences formed by connectives are also called **compound sentences** or **compound statements**. One of the simplest compound statements that can be formed uses the connective *and*.

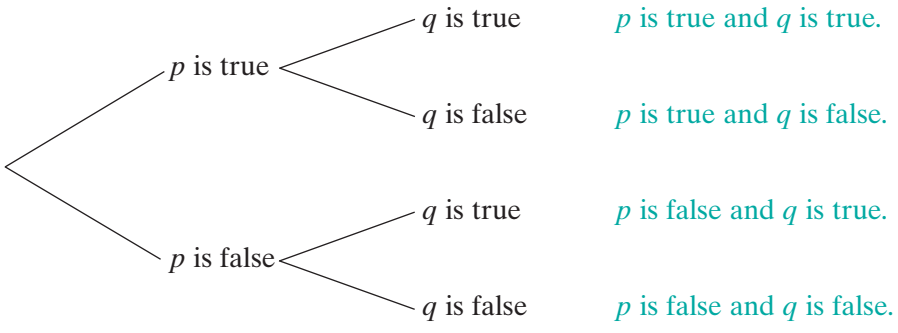
In logic, a **conjunction** is a compound statement formed by combining two simple statements using the word *and*. Each of the simple statements is called a **conjunct**. When p and q represent simple statements, the conjunction ***p* and *q*** is written in symbols as $\boldsymbol{p \wedge q}$. For example:

p: A dog is an animal.
q: A sparrow is a bird.
$p \wedge q$: A dog is an animal and a sparrow is a bird.

This compound sentence is true because both parts are true. "A dog is an animal is true and "A sparrow is a bird" is true. When one or both parts of a conjunction are false, the conjunction is false. For example,

- "A dog is an animal and a sparrow is not a bird" is false because "A sparrow is not a bird" is the negation of a true statement and is false.
- "A dog is not an animal and a sparrow is a bird" is false because "A dog is not an animal" is the negation of a true statement and is false.
- "A dog is not an animal and a sparrow is not a bird" is false because both "A dog is not an animal" and "A sparrow is not a bird" are false.

We can draw a diagram, called a **tree diagram**, to show all possible combinations of the truth values of p and q that are combined to make the compound statement $p \wedge q$.



These four possible combinations of the truth values of p and q can be displayed in a chart called a **truth table**. The truth table can be used to show the possible truth values of a compound statement that is made up of two simple statements.

For instance, write a truth table for $p \wedge q$.

STEP 1. In the first column, we list the truth values of p. For each possible truth value of p, there are two possible truth values for q. Therefore, we list T twice and F twice.

p		
T		
T		
F		
F		

STEP 2. In the second column, we list the truth values of q. In the two rows in which p is true, list q as true in one and false in the other. In the two rows in which p is false, list q as true in one and false in the other.

p	q	
T	T	
T	F	
F	T	
F	F	

STEP 3. In the last column, list the truth values of the conjunction, $p \wedge q$. The conjunction is true only when both p and q are true. The conjunction is false when one or both conjuncts are false.

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

► **The conjunction, p and q, is true only when both parts are true: p must be true and q must be true.**

For example, let p represent "It is spring," and let q represent "It is March."

CASE 1 *Both p and q are true.*

On March 30, "It is spring" is true and "It is March" is true. Therefore, "It is spring and it is March" is true.

CASE 2 *p is true and q is false.*

On April 30, "It is spring" is true and "It is March" is false. Therefore, "It is spring and it is March" is false.

CASE 3 *p is false and q is true.*

On March 10, "It is spring" is false and "It is March" is true. Therefore, "It is spring and it is March" is false.

CASE 4 *Both p and q are false.*

On February 28, "It is spring" is false and "It is March" is false. Therefore, "It is spring and it is March" is false.

A compound sentence may contain both negations and conjunctions at the same time. For example:

Let p represent "Ten is divisible by 2."
Let q represent "Ten is divisible by 3."

Then $p \wedge \sim q$ represents "Ten is divisible by 2 and ten is not divisible by 3." Here p is true, q is false, and $\sim q$ is true. Then for $p \wedge \sim q$, both parts are true so the conjunction is true. This can be summarized in the following table.

p	q	$\sim q$	$p \wedge \sim q$
T	F	T	T

EXAMPLE 1

Let p represent "Albany is the capital of New York State." (True)

Let q represent "Philadelphia is the capital of Pennsylvania." (False)

For each given sentence:

a. Write the sentence in symbolic form.

b. Tell whether the statement is true or false.

(1) Albany is the capital of New York State and Philadelphia is the capital of Pennsylvania.

(2) Albany is the capital of New York State and Philadelphia is not the capital of Pennsylvania.

(3) Albany is not the capital of New York State and Philadelphia is the capital of Pennsylvania.

(4) Albany is not the capital of New York State and Philadelphia is not the capital of Pennsylvania.

(5) It is not true that Albany is the capital of New York State and Philadelphia is the capital of Pennsylvania.

Solution (1) The statement is a conjunction. Since p is true and q is false, the statement is false.

p	q	$p \wedge q$
T	F	F

Answers: **a.** $p \wedge q$ **b.** False

(2) The statement is a conjunction. Since p is true and $\sim q$ is true, the statement is true.

p	q	$\sim q$	$p \wedge \sim q$
T	F	T	T

Answers: **a.** $p \wedge \sim q$ **b.** True

(3) The statement is a conjunction. Since $\sim p$ is false and q is false, the statement is false.

p	$\sim p$	q	$\sim p \wedge q$
T	F	F	F

Answers: **a.** $\sim p \wedge q$ **b.** False

(4) The statement is a conjunction. Since $\sim p$ is false and $\sim q$ is true, the statement is false.

p	$\sim p$	q	$\sim q$	$\sim p \wedge \sim q$
T	F	F	T	F

Answers: **a.** $\sim p \wedge \sim q$ **b.** False

(5) The phrase "It is not true that" applies to the entire conjunction. Since p is true and q is false, $(p \wedge q)$ is false and the negation of $(p \wedge q)$ is true.

p	q	$p \wedge q$	$\sim(p \wedge q)$
T	F	F	T

Answers: **a.** $\sim(p \wedge q)$ **b.** True

EXAMPLE 2

Use the domain $\{1, 2, 3, 4\}$ to find the truth set for the open sentence

$$(x < 3) \wedge (x \text{ is a prime})$$

Solution

Let $x = 1$	$(1 < 3) \wedge (1 \text{ is a prime})$	$\text{T} \wedge \text{F} = \text{False}$
Let $x = 2$	$(2 < 3) \wedge (2 \text{ is a prime})$	$\text{T} \wedge \text{T} = \text{True}$
Let $x = 3$	$(3 < 3) \wedge (3 \text{ is a prime})$	$\text{F} \wedge \text{T} = \text{False}$
Let $x = 4$	$(4 < 3) \wedge (4 \text{ is a prime})$	$\text{F} \wedge \text{F} = \text{False}$

A conjunction is true only when both simple sentences are true. This condition is met here when $x = 2$. Thus, the truth set or solution set is $\{2\}$.

Answer $\{2\}$

EXAMPLE 3

Three sentences are written below. The truth values are given for the first two sentences. Determine whether the third sentence is true, is false, or has an uncertain truth value.

Today is Friday and I have soccer practice.	(False)
Today is Friday.	(True)
I have soccer practice.	(?)

Solution Since the conjunction is false, at least one of the conjuncts must be false. But "Today is Friday" is true. Therefore, "I have soccer practice" must be false.

Answer "I have soccer practice" is false.

EXAMPLE 4

Three sentences are written below. The truth values are given for the first two sentences. Determine whether the third sentence is true, is false, or has an uncertain truth value.

Today is Monday and the sun is shining.	(False)
Today is Monday.	(False)
The sun is shining.	(?)

Solution (1) Use symbols to represent the sentences. Indicate their truth values.

p:	Today is Monday.	(False)
q:	The sun is shining.	(?)
$p \wedge q$:	Today is Monday and the sun is shining.	(False)

(2) Construct a truth for the conjunction. Study the truth values of p and $p \wedge q$.
Since $p \wedge q$ is false, the last three rows apply. Since p is false, the choices are narrowed to the last two rows. Therefore q could be either true or false.

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Answer The truth value of "The sun is shining" is uncertain.

Exercises

Writing About Mathematics

1. Is the negation of a conjunction, $\sim(p \wedge q)$, the same as $\sim p \wedge \sim q$? Justify your answer.

2. What must be the truth values of p, q, and r in order for $(p \wedge q) \wedge r$ to be true? Explain your answer.

Developing Skills

In 3–12, write each sentence in symbolic form, using the given symbols.

Let p represent "It is hot."
Let q represent "It is raining."
Let r represent "The sky is cloudy."

3. It is hot and it is raining.

4. It is hot and the sky is cloudy.

5. It is not hot.

6. It is not hot and the sky is cloudy.

7. It is raining and the sky is not cloudy.

8. It is not hot and it is not raining.

9. The sky is not cloudy and it is not hot.

10. The sky is not cloudy and it is hot.

11. It is not the case that it is hot and it is raining.

12. It is not the case that it is raining and it is not hot.

In 13–20, using the truth value for each given statement, tell if the conjunction is true or false.

A piano is a percussion instrument.	(True)
A piano has 88 keys.	(True)
A flute is a percussion instrument.	(False)
A trumpet is a brass instrument.	(True)

13. A flute is a percussion instrument and a piano is a percussion instrument.

14. A flute is a percussion instrument and a trumpet is a brass instrument.

15. A piano has 88 keys and is a percussion instrument.

16. A piano has 88 keys and a trumpet is a brass instrument.

17. A piano is not a percussion instrument and a piano does not have 88 keys.

18. A trumpet is not a brass instrument and a piano is a percussion instrument.

19. A flute is not a percussion instrument and a trumpet is a brass instrument.

20. It is not true that a piano is a percussion instrument and has 88 keys.

In 21–28, complete each sentence with "true" or "false" to make a correct statement.

21. When p is true and q is true, then $p \wedge q$ is _____.

22. When p is false, then $p \wedge q$ is _____.

23. If p is true, or q is true, but not both, then $p \wedge q$ is _____.

24. When $p \wedge q$ is true, then p is _____ and q is _____.

25. When $p \wedge \sim q$ is true, then p is _____ and q is _____.

26. When $\sim p \wedge q$ is true, then p is _____ and q is _____.

27. When p is false and q is true, then $\sim(p \wedge q)$ is _____.

28. If both p and q are false, then $\sim p \wedge \sim q$ is _____.

Applying Skills

In 29–36, three sentences are written. The truth values are given for the first two sentences. Determine whether the third sentence is true, is false, or has an uncertain truth value.

29. It is noon and I get lunch. (True)
It is noon. (True)
I get lunch. (?)

30. I have the hiccups and I drink some water. (False)
I have the hiccups. (True)
I drink some water. (?)

31. I have the hiccups and I drink some water. (False)
I have the hiccups. (False)
I drink some water. (?)

32. I play tennis and Anna plays golf. (False)
I play tennis. (True)
Anna plays golf. (?)

33. Pam sees a movie and Pam loves going to the theater. (True)
Pam sees a movie. (True)
Pam loves going to the theater. (?)

34. Jon and Edith like to eat ice cream. (False)
Jon likes to eat ice cream. (True)
Edith likes to eat ice cream. (?)

35. Jordan builds model trains and model planes. (False)
Jordan builds model trains. (False)
Jordan builds model planes. (?)

36. Bethany likes to chat and surf on the internet. (False)
Bethany likes to surf on the internet. (True)
Bethany likes to chat on the internet. (?)

In 37 and 38, a compound sentence is given using a conjunction. Use the truth value of the compound sentence to determine whether each sentence that follows is true or false.

37. In winter I wear a hat and scarf. (True)

a. In winter I wear a hat.

b. In winter I wear a scarf.

c. In winter I do not wear a hat.

38. I do not practice and I know that I should. (True)

a. I do not practice.

b. I know that I should practice.

c. I practice.

2-3 DISJUNCTIONS

In logic, a **disjunction** is a compound statement formed by combining two simple statements using the word *or*. Each of the simple statements is called a **disjunct**. When p and q represent simple statements, the disjunction ***p* or *q*** is written in symbols as $p \vee q$. For example:

p: Andy rides his bicycle to school.
q: Andy walks to school.
$p \vee q$: Andy rides his bicycle to school or Andy walks to school.

In this example, when is the disjunction $p \vee q$ true and when is it false?

1. On Monday, Andy rode his bicycle part of the way to school when he met a friend. Then he walked the rest of the way to school with his friend. Here p is true and q is true. The disjunction $p \vee q$ is true.

2. On Tuesday, Andy rode his bicycle to school and did not walk to school. Here p is true and q is false. The disjunction $p \vee q$ is true.

3. On Wednesday, Andy did not ride his bicycle to school and walked to school. Here p is false and q is true. The disjunction $p \vee q$ is true.

4. On Thursday, it rained so Andy's father drove him to school. Andy did not ride his bicycle to school and did not walk to school. Here p is false and q is false. The disjunction $p \vee q$ is false.

► **The disjunction p or q is true when any part of the compound sentence is true: p is true, q is true, or both p and q are true.**

The only case in which the disjunction p *or* q is false is when both p and q are false. The truth values of the disjunction $p \vee q$ are summarized in the truth table to the right. The possible combinations of the truth values of p and of q, shown in the first two columns, are the same as those used in the truth table of the conjunction. The third column gives the truth values for the disjunction, $p \vee q$.

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

EXAMPLE 1

Use the following statements:

Let k represent "Kurt plays baseball."
Let a represent "Alicia plays baseball."
Let n represent "Nathan plays soccer."

Write each given sentence in symbolic form.

	Answers
a. Kurt or Alicia play baseball.	$k \vee a$
b. Kurt plays baseball or Nathan plays soccer.	$k \vee n$
c. Alicia plays baseball or Alicia does not play baseball.	$a \vee \sim a$
d. It is not true that Kurt or Alicia play baseball.	$\sim(k \vee a)$
e. Either Kurt does not play baseball or Alicia does not play baseball.	$\sim k \vee \sim a$
f. It's not the case that Alicia or Kurt play baseball.	$\sim(a \vee k)$

EXAMPLE 2

Symbols are used to represent three statements. For each statement, the truth value is noted.

k:	"Every line segment has a midpoint."	(True)
m:	"A line has a midpoint."	(False)
q:	"A ray has one endpoint."	(True)

For each sentence given in symbolic form:

a. Write a complete sentence in words to show what the symbols represent.

b. Tell whether the statement is true or false.

	Answers
(1) $k \vee q$	**a.** Every line segment has a midpoint or every ray has one endpoint. **b.** $\mathrm{T} \vee \mathrm{T}$ is a true disjunction.
(2) $k \vee m$	**a.** Every line segment has a midpoint or a line has a midpoint. **b.** $\mathrm{T} \vee \mathrm{F}$ is a true disjunction.
(3) $m \vee \sim q$	**a.** A line has a midpoint or a ray does not have one endpoint. **b.** $\mathrm{F} \vee \sim\mathrm{T} = \mathrm{F} \vee \mathrm{F}$, a false disjunction.
(4) $\sim(m \vee q)$	**a.** It is not the case that a line has a midpoint or a ray has one endpoint. **b.** $\sim(\mathrm{F} \vee \mathrm{T}) = \sim\mathrm{T}$, a false disjunction.

EXAMPLE 3

Find the solution set of each of the following if the domain is the set of positive integers less than 8.

a. $(x < 4) \vee (x > 3)$ **b.** $(x > 3) \vee (x \text{ is odd})$ **c.** $(x > 5) \wedge (x < 3)$

Solution The domain is $\{1, 2, 3, 4, 5, 6, 7\}$.

a. The solution set of $x < 4$ is $\{1, 2, 3\}$ and the solution set of $x > 3$ is $\{4, 5, 6, 7\}$. The solution set of the disjunction $(x < 4) \vee (x > 3)$ includes all the numbers that make $x < 4$ true together with all the numbers that make $x > 3$ true.

Answer $\{1, 2, 3, 4, 5, 6, 7\}$

Note: The solution set of the disjunction $(x < 4) \vee (x > 3)$ is the union of the solution sets of the individual parts: $\{1, 2, 3\} \cup \{4, 5, 6, 7\} = \{1, 2, 3, 4, 5, 6, 7\}$.

b. The solution set of $(x > 3)$ is $\{4, 5, 6, 7\}$ and the solution set of $(x \text{ is odd})$ is $\{1, 3, 5, 7\}$. The solution set of the disjunction $(x > 3) \vee (x \text{ is odd})$ includes all the numbers that make either $x > 3$ true or x is odd true.

Answer $\{1, 3, 4, 5, 6, 7\}$

Note: The solution set of the disjunction $(x > 3) \vee (x \text{ is odd})$ is the union of the solution sets: $\{4, 5, 6, 7\} \cup \{1, 3, 5, 7\} = \{1, 3, 4, 5, 6, 7\}$.

c. The solution set of $(x > 5)$ is $\{6, 7\}$ and the solution set of $(x < 3)$ is $\{1, 2\}$. The solution set of $(x > 5) \wedge (x < 3)$ is $\{6, 7\} \cap \{1, 2\}$ or the empty set, $\varnothing$.

Answer $\varnothing$

Two Uses of the Word *Or*

When we use the word *or* to mean that *one* or *both* of the simple sentences are true, we call this the **inclusive *or***. The truth table we have just shown uses truth values for the *inclusive or*.

Sometimes, however, the word *or* is used in a different way, as in "He is in grade 9 or he is in grade 10." Here it is not possible for both simple sentences to be true at the same time. When we use the word *or* to mean that *one and only one* of the simple sentences is true, we call this the **exclusive *or***. The truth table for the *exclusive or* will be different from the table shown for disjunction. In the *exclusive or*, the disjunction *p or q* will be true when *p* is true, or when *q* is true, but not both.

In everyday conversation, it is often evident from the context which of these uses of *or* is intended. In legal documents or when ambiguity can cause difficulties, the *inclusive or* is sometimes written as *and/or*.

► **We will use only the *inclusive or* in this book. Whenever we speak of disjunction, *p or q* will be true when *p* is true, when *q* is true, when both *p* and *q* are true.**

Exercises

Writing About Mathematics

1. Explain the relationship between the truth set of the negation of a statement and the complement of a set.

2. Explain the difference between the *inclusive or* and the *exclusive or*.

Developing Skills

In 3–12, for each given statement: **a.** Write the statement in symbolic form, using the symbols given below. **b.** Tell whether the statement is true or false.

Let *c* represent "A gram is 100 centigrams."	(True)
Let *m* represent "A gram is 1,000 milligrams."	(True)
Let *k* represent "A kilogram is 1,000 grams."	(True)
Let *l* represent "A gram is a measure of length."	(False)

3. A gram is 1,000 milligrams or a kilogram is 1,000 grams.

4. A gram is 100 centigrams or a gram is a measure of length.

5. A gram is 100 centigrams or 1,000 milligrams.

6. A kilogram is not 1,000 grams or a gram is not 100 centigrams.

7. A gram is a measure of length or a kilogram is 1,000 grams.

8. A gram is a measure of length and a gram is 100 centigrams.

9. It is not the case that a gram is 100 centigrams or 1,000 milligrams.

10. It is false that a kilogram is not 1,000 grams or a gram is a measure of length.

11. A gram is 100 centigrams and a kilogram is 1,000 grams.

12. A gram is not 100 centigrams or is not 1,000 milligrams, and a gram is a measure of length.

In 13–20, symbols are assigned to represent sentences.

Let b represent "Breakfast is a meal."
Let s represent "Spring is a season."
Let h represent "Halloween is a season."

For each sentence given in symbolic form: **a.** Write a complete sentence in words to show what the symbols represent. **b.** Tell whether the sentence is true or false.

13. $s \vee h$ **14.** $b \wedge s$ **15.** $\sim s \vee h$ **16.** $b \wedge \sim h$

17. $\sim b \vee \sim s$ **18.** $\sim(s \wedge h)$ **19.** $\sim(b \vee \sim s)$ **20.** $\sim b \wedge \sim s$

In 21–27, complete each sentence with the words "true" or "false" to make a correct statement.

21. When p is true, then $p \vee q$ is _____.

22. When q is true, then $p \vee q$ is _____.

23. When p is false and q is false, then $p \vee q$ is _____.

24. When $p \vee \sim q$ is false, then p is _____ and q is _____.

25. When $\sim p \vee q$ is false, then p is _____ and q is _____.

26. When p is false and q is true, then $\sim(p \vee q)$ is _____.

27. When p is false and q is true, then $\sim p \vee \sim q$ is _____.

Applying Skills

In 28–32, three sentences are written. The truth values are given for the first two sentences. Determine whether the third sentence is true, is false, or has an uncertain truth value.

28. May is the first month of the year. (False)
January is the first month of the year. (True)
May is the first month of the year or January is the first month of the year. (?)

29. I will study more or I will fail the course. (True)
I will fail the course. (False)
I will study more. (?)

30. Jen likes to play baseball and Mason likes to play baseball. (False)
Mason likes to play baseball. (True)
Jen likes to play baseball. (?)

31. Nicolette is my friend or Michelle is my friend. (True)
Nicolette is my friend. (True)
Michelle is my friend. (?)

32. I practice the cello on Monday or I practice the piano on Monday. (True)
I do not practice the piano on Monday. (False)
I practice the cello on Monday. (?)

2-4 CONDITIONALS

A sentence such as "If I have finished my homework, then I will go to the movies" is frequently used in daily conversation. This statement is made up of two simple statements:

p: I have finished my homework.
q: I will go to the movies.

The remaining words, *if . . . then,* are the connectives.

In English, this sentence is called a complex sentence. In mathematics, however, all sentences formed using connectives are called *compound sentences* or *compound statements*.

In logic, a **conditional** is a compound statement formed by using the words *if . . . then* to combine two simple statements. When p and q represent simple statements, the conditional **if p then q** is written in symbols as $p \to q$. The symbol $p \to q$ can also be read as "p implies q" or as "p only if q."

Here is another example:

p: It is January.
q: It is winter.
$p \to q$: If it is January, then it is winter.
or
It is January implies that it is winter.
or
It is January only if it is winter.

Certainly we would agree that the compound sentence *if p then q* is true for this example: "If it is January, then it is winter." However, if we reverse the order of the simple sentences to form the new conditional *if q then p*, we will get a sentence with a different meaning:

$q \to p$: If it is winter, then it is January.

When it is winter, it does not necessarily mean that it is January. It may be February, the last days of December, or the first days of March. Changing the order in which we connect two simple statements in conditional does not always give a conditional that has the same truth value as the original.

Parts of a Conditional

The parts of the conditional *if p then q* can be identified by name:

p is the **hypothesis**, which is sometimes referred to as the **premise** or the **antecedent**. It is an assertion or a sentence that begins an argument. The hypothesis usually follows the word *if*.

q is the **conclusion**, which is sometimes referred to as the **consequent**. It is the part of a sentence that closes an argument. The conclusion usually follows the word *then*.

There are different ways to write the conditional. When the conditional uses the word *if*, the hypothesis always follows *if*. When the conditional uses the word *implies*, the hypothesis always comes before *implies*. When the conditional uses the words *only if*, the conclusion follows the words *only if*.

$p \to q$: If $\underbrace{\text{it is January}}_{\text{hypothesis}}$, then $\underbrace{\text{it is winter}}_{\text{conclusion}}$.

$p \to q$: $\underbrace{\text{It is January}}_{\text{hypothesis}}$ implies that $\underbrace{\text{it is winter}}_{\text{conclusion}}$.

$p \to q$: $\underbrace{\text{It is January}}_{\text{hypothesis}}$ only if $\underbrace{\text{it is winter.}}_{\text{conclusion}}$

All three sentences say the same thing. We are able to draw a conclusion about the season when we know that the month is January. Although the word order of a conditional may vary, the hypothesis is always written first when using symbols.

Truth Values for the Conditional $p \to q$

In order to determine the truth value of a conditional, we will consider the statement "If you get an A in Geometry, then I will buy you a new graphing calculator." Let p represent the hypothesis, and let q represent the conclusion.

p: You get an A in Geometry.
q: I will buy you a new graphing calculator.

Determine the truth values of the conditional by considering all possible combinations of the truth values for p and q.

CASE 1

You get an A in Geometry. (p is true.)
I buy you a new graphing calculator. (q is true.)
We both keep our ends of the agreement.
The conditional statement is true.

p	q	$p \to q$
T	T	T

CASE 2

p	q	$p \to q$
T	F	F

You get an A in Geometry. (p is true.)
I do not buy you a new graphing calculator. (q is false.)
I broke the agreement because you got an A in Geometry but I did *not* buy you a new graphing calculator.
The conditional statement is false.

CASE 3

p	q	$p \to q$
F	T	T

You do not get an A in Geometry. (p is false.)
I buy you a new graphing calculator. (q is true.)
You did *not* get an A but I bought you a new graphing calculator anyway. Perhaps I felt that a new calculator would help you to get an A next time. I did not break my promise. My promise only said what I would do if you did get an A.
The conditional statement is true.

CASE 4

p	q	$p \to q$
F	F	T

You do not get an A in Geometry. (p is false.)
I do not buy you a new graphing calculator. (q is false.)
Since you did not get an A, I do not have to keep our agreement.
The conditional statement is true.

Case 2 tells us that the conditional is false only when the hypothesis is true and the conclusion is false. If a conditional is thought of as an "agreement" or a "promise," this corresponds to the case when the agreement is broken.

Cases 3 and 4 tell us that when the hypothesis is false, the conclusion may or may not be true. In other words, if you do not get an A in Geometry, I may or may not buy you a new graphing calculator.

These four cases can be summarized as follows:

► **A conditional is false when a true hypothesis leads to a false conclusion. In all other cases, the conditional is true.**

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

Hidden Conditionals

Often the words "if . . . then" may not appear in a statement that does suggest a conditional. Instead, the expressions "when" or "in order that" may suggest that the statement is a conditional. For example:

1. "When I finish my homework I will go to the movies."
$p \to q$: *If* I finish my homework, *then* I will go to the movies.

2. "In order to succeed, you must work hard" becomes
$p \to q$: *If* you want to succeed, *then* you must work hard.

3. "$2x = 10$; therefore $x = 5$" becomes
$p \to q$: *If* $2x = 10$, *then* $x = 5$.

EXAMPLE 1

For each given sentence:

a. Identify the hypothesis p. **b.** Identify the conclusion q.

(1) If Mrs. Shusda teaches our class, then we will learn.

(2) The assignment will be completed if I work at it every day.

(3) The task is easy when we all work together and do our best.

Solution (1) If $\underbrace{\text{Mrs. Shusda teaches our class}}_{p}$, then $\underbrace{\text{we will learn}}_{q}$.

a. p: Mrs. Shusda teaches our class.

b. q: We will learn.

(2) $\underbrace{\text{The assignment will be completed}}_{q}$ if $\underbrace{\text{I work at it every day}}_{p}$.

a. p: I work at it every day.

b. q: The assignment will be completed.

(3) *Hidden Conditional:*

If $\underbrace{\text{we all work together and do our best}}_{p}$, then $\underbrace{\text{the task is easy}}_{q}$.

a. p: We all work together and we do our best.

b. q: the task is easy.

Note: In (3), the hypothesis is a conjunction. If we let r represent "We all work together" and s represent "We do our best," then the conditional "If we all work together and we do our best, then the task is easy" can be symbolized as $(r \wedge s) \to q$.

EXAMPLE 2

Identify the truth value to be assigned to each conditional statement.

(1) If $4 + 4 = 8$, then $2(4) = 8$.

(2) If 2 is a prime number, then 2 is odd.

(3) If 12 is a multiple of 9, then 12 is a multiple of 3.

(4) If $2 > 3$ then $2 - 3$ is a positive integer.

Solution (1) The hypothesis p is "$4 + 4 = 8$," which is true.
The conclusion q is "$2(4) = 8$," which is true.
The conditional $p \to q$ is true. *Answer*

(2) The hypothesis p is "2 is a prime number," which is true.
The conclusion q is "2 is odd," which is false.
The conditional $p \to q$ is false. *Answer*

(3) The hypothesis p is "12 is a multiple of 9," which is false.
The conclusion q is "12 is a multiple of 3," which is true.
The conditional $p \to q$ is true. *Answer*

(4) The hypothesis p is "$2 > 3$," which is false.
The conclusion q is "$2 - 3$ is a positive integer," which is false.
The conditional $p \to q$ is true. *Answer*

EXAMPLE 3

For each given statement:

a. Write the statement in symbolic form using the symbols given below.

b. Tell whether the statement is true or false.

Let m represent "Monday is the first day of the week." (True)
Let w represent "There are 52 weeks in a year." (True)
Let h represent "An hour has 75 minutes." (False)

Answers

(1) If Monday is the first day of the week, then there are 52 weeks in a year.

a. $m \to w$

b. $T \to T$ is true.

(2) If there are 52 weeks in a year, then an hour has 75 minutes.

a. $w \to h$

b. $T \to F$ is false.

(3) If there are not 52 weeks in a year then Monday is the first day of the week.

a. $\sim w \to m$

b. $F \to T$ is true.

(4) If Monday is the first day of the week and there are 52 weeks in a year, then an hour has 75 minutes.

a. $(m \wedge w) \to h$

b. $(T \wedge T) \to F$

$T \to F$ is false.

Exercises

Writing About Mathematics

1. a. Show that the conditional "If x is divisible by 4, then x is divisible by 2" is true in each of the following cases:

(1) $x = 8$ (2) $x = 6$ (3) $x = 7$

b. Is it possible to find a value of x for which the hypothesis is true and the conclusion is false? Explain your answer.

2. For what truth values of p and q is the truth value of $p \rightarrow q$ the same as the truth value of $q \rightarrow p$?

Developing Skills

In 3–10, for each given sentence: **a.** Identify the hypothesis p. **b.** Identify the conclusion q.

3. If a polygon is a square, then it has four right angles.

4. If it is noon, then it is time for lunch.

5. When you want help, ask a friend.

6. You will finish more quickly if you are not interrupted.

7. The perimeter of a square is $4s$ if the length of one side is s.

8. If many people work at a task, it will be completed quickly.

9. $2x + 7 = 11$ implies that $x = 2$.

10. If you do not get enough sleep, you will not be alert.

In 11–16, write each sentence in symbolic form, using the given symbols.

p: The car has a flat tire.
q: Danny has a spare tire.
r: Danny will change the tire.

11. If the car has a flat tire, then Danny will change the tire.

12. If Danny has a spare tire, then Danny will change the tire.

13. If the car does not have a flat tire, then Danny will not change the tire.

14. Danny will not change the tire if Danny doesn't have a spare tire.

15. The car has a flat tire if Danny has a spare tire.

16. Danny will change the tire if the car has a flat tire.

In 17–24, for each given statement: **a.** Write the statement in symbolic form, using the symbols given below. **b.** Tell whether the conditional statement is true or false, based upon the truth values given.

b:	The barbell is heavy.	(True)
t:	Kylie trains.	(False)
l:	Kylie lifts the barbell.	(True)

17. If Kylie trains, then Kylie will lift the barbell.

18. If Kylie lifts the barbell, then Kylie has trained.

19. If Kylie lifts the barbell, the barbell is heavy.

20. Kylie lifts the barbell if the barbell is not heavy.

21. Kylie will not lift the barbell if Kylie does not train.

22. Kylie trains if the barbell is heavy.

23. If the barbell is not heavy and Kylie trains, then Kylie will lift the barbell.

24. If the barbell is heavy and Kylie does not train, then Kylie will not lift the barbell.

In 25–31, find the truth value to be assigned to each conditional statement.

25. If $4 + 8 = 12$, then $8 + 4 = 12$.

26. If $9 < 15$, then $19 < 25$.

27. If $1 \times 1 = 1$, then $1 \times 1 \times 1 = 1$.

28. $24 \div 3 = 8$ if $24 \div 8 = 3$.

29. $6 + 6 = 66$ if $7 + 7 = 76$.

30. $48 = 84$ if $13 = 31$.

31. If every rhombus is a polygon, then every polygon is a rhombus.

In 32–39, symbols are assigned to represent sentences, and truth values are assigned to these sentences.

Let j represent "July is a warm month."	(True)
Let d represent "I am busy every day."	(False)
Let g represent "I work in my garden."	(True)
Let f represent "I like flowers."	(True)

For each compound statement in symbolic form: **a.** Write a complete sentence in words to show what the symbols represent. **b.** Tell whether the compound statement is true or false.

32. $j \to g$

33. $d \to \sim g$

34. $f \to g$

35. $\sim g \to \sim j$

36. $(j \wedge f) \to d$

37. $(j \wedge g) \to f$

38. $\sim j \to (d \wedge f)$

39. $g \to (j \vee \sim d)$

In 40–45 supply the word, phrase, or symbol that can be placed in the blank to make each resulting sentence true.

40. When p and q represent two simple sentences, the conditional *if p then q* is written symbolically as _____.

41. The conditional *if q then p* is written symbolically as _____.

42. The conditional $p \to q$ is false only when p is _____ and q is _____.

43. When the conclusion q is true, then $p \to q$ must be _____.

44. When the hypothesis p is false, then $p \to q$ must be _____.

45. If the hypothesis p is true and conditional $p \to q$ is true, then the conclusion q must be _____.

Applying Skills

In 46–50, three sentences are written in each case. The truth values are given for the first two sentences. Determine whether the third sentence is true, is false, or has an uncertain truth value.

46. If you read in dim light, then you can strain your eyes. (True)
You read in dim light. (True)
You can strain your eyes. (?)

47. If the quadrilateral has four right angles, then the quadrilateral must be a square. (False)
The quadrilateral has four right angles. (True)
The quadrilateral must be a square. (?)

48. If n is an odd number, then $2n$ is an even number. (True)
$2n$ is an even number. (True)
n is an odd number. (?)

49. If the report is late, then you will not get an A. (True)
The report is late. (False)
You will not get an A. (?)

50. Area $= \frac{1}{2}bh$ if the polygon is a triangle. (True)
The polygon is a triangle. (True)
Area $= \frac{1}{2}bh$ (?)

2-5 INVERSES, CONVERSES, AND CONTRAPOSITIVES

The conditional is the most frequently used statement in the construction of an argument or in the study of mathematics. We will use the conditional frequently in our study of geometry. In order to use the conditional statements correctly, we must understand their different forms and how their truth values are related.

There are four conditionals that can be formed from two simple statements, p and q, and their negations.

The conditional:	$p \to q$	The inverse:	$\sim p \to \sim q$
The converse:	$q \to p$	The contrapositive:	$\sim q \to \sim p$

The Inverse

The **inverse** of a conditional statement is formed by negating the hypothesis and the conclusion. For example, the inverse of the statement "If today is Monday, then I have soccer practice" is "If today is *not* Monday, then I do *not* have soccer practice." In symbols, the inverse of $(p \to q)$ is $(\sim p \to \sim q)$.

The following examples compare the truth values of given conditionals and their inverses.

1. *A true conditional can have a false inverse.*

Let p represent "A number is divisible by ten."
Let q represent "The number is divisible by five."

Conditional ($p \to q$): If $\underbrace{\text{a number is divisible by ten}}_{p}$, then $\underbrace{\text{it is divisible by five}}_{q}$.

Inverse ($\sim p \to \sim q$): If $\underbrace{\text{a number is } \textit{not} \text{ divisible by ten}}_{\sim p}$, then $\underbrace{\text{it is } \textit{not} \text{ divisible by five}}_{\sim q}$.

We can find the truth value of these two statements when the number is 15.

p:	15 is divisible by 10.	False
q:	15 is divisible by 5.	True
$p \to q$:	If 15 is divisible by 10, then 15 is divisible by 5.	F $\to$ T is T
$\sim p$:	15 is not divisible by 10.	True
$\sim q$:	15 is not divisible by 5.	False
$\sim p \to \sim q$:	If 15 is not divisible by 10, then 15 is not divisible by 5.	T $\to$ F is F

In this case, the conditional and the inverse have opposite truth values.

2. *A false conditional can have a true inverse.*

Let p represent "Two angles are congruent."
Let q represent "Two angles are both right angles."

Conditional ($p \to q$): If $\underbrace{\text{two angles are congruent}}_{p}$,
then $\underbrace{\text{the two angles are both right angles}}_{q}$.

Inverse ($\sim p \to \sim q$): If $\underbrace{\text{two angles are not congruent}}_{\sim p}$,
then $\underbrace{\text{the two angles are not both right angles}}_{\sim q}$.

We can find the truth value of these two statements with two angles A and B when $m\angle A = 60$ and $m\angle B = 60$.

p:	Angle A and $\angle B$ are congruent.	True
q:	Angle A and $\angle B$ are both right angles.	False
$p \to q$:	If $\angle A$ and $\angle B$ are congruent, then $\angle A$ and $\angle B$ are both right angles.	$T \to F$ is F
$\sim p$:	Angle A and $\angle B$ are not congruent.	False
$\sim q$:	Angle A and $\angle B$ are not both right angles.	True
$\sim p \to \sim q$:	If $\angle A$ and $\angle B$ are not congruent, then $\angle A$ and $\angle B$ are not both right angles.	$F \to T$ is T

Again, the conditional and the inverse have opposite truth values.

3. *A conditional and its inverse can have the same truth value.*

Let r represent "Twice Talia's age is 10."
Let q represent "Talia is 5 years old."

Conditional ($r \to s$): If $\underbrace{\text{twice Talia's age is 10}}_{r}$, then $\underbrace{\text{Talia is 5 years old}}_{s}$.

Inverse ($\sim r \to \sim s$): If $\underbrace{\text{twice Talia's age is } \textit{not} \text{ 10}}_{\sim r}$, then $\underbrace{\text{Talia is } \textit{not} \text{ 5 years old}}_{\sim s}$.

When Talia is 5, r is true and s is true. The conditional is true.
When Talia is 5, $\sim r$ is false and $\sim s$ is false. The inverse is true.
Both the conditional and its inverse have the same truth value.

When Talia is 6, r is false and s is false. The conditional is true.
When Talia is 6, $\sim r$ is true and $\sim s$ is true. The conditional is true.
Again, the conditional and its inverse have the same truth value.

These three illustrations allow us to make the following conclusion:

► **A conditional ($p \to q$) and its inverse ($\sim p \to \sim q$) may or may not have the same truth value.**

This conclusion can be shown in the truth table. Note that the conditional ($p \to q$) and its inverse ($\sim p \to \sim q$) have the same truth value when p and q have the same truth value. The conditional ($p \to q$) and its inverse ($\sim p \to \sim q$) have opposite truth values when p and q have opposite truth values.

				Conditional	Inverse
p	q	$\sim p$	$\sim q$	$p \to q$	$\sim p \to \sim q$
T	T	F	F	T	T
T	F	F	T	F	T
F	T	T	F	T	F
F	F	T	T	T	T

The Converse

The **converse** of a conditional statement is formed by interchanging the hypothesis and conclusion. For example, the converse of the statement "If today is Monday, then I have soccer practice" is "If I have soccer practice, then today is Monday." In symbols, the converse of $(p \to q)$ is $(q \to p)$.

To compare the truth values of a conditional and its converse, we will consider some examples.

1. *A true conditional can have a false converse.*

Let p represent "x is a prime."
Let q represent "x is odd."

Conditional ($p \to q$): If $\underbrace{x \text{ is a prime}}_{p}$, then $\underbrace{x \text{ is odd}}_{q}$.

Converse ($q \to p$): If $\underbrace{x \text{ is odd}}_{q}$, then $\underbrace{x \text{ is a prime}}_{p}$.

When $x = 9$, p is false and q is true. Therefore, for this value of x, the conditional $(p \to q)$ is true and its converse $(q \to p)$ is false.

In this example, the conditional is true and the converse is false. The conditional and its converse do not have the same truth value.

2. *A false conditional can have a true converse.*

Let p represent "x is divisible by 2."
Let q represent "x is divisible by 6."

Conditional ($p \to q$): If $\underbrace{x \text{ is divisible by } 2}_{p}$, then $\underbrace{x \text{ is divisible by } 6}_{q}$.

Converse ($q \to p$): If $\underbrace{x \text{ is divisible by } 6}_{q}$, then $\underbrace{x \text{ is divisible by } 2}_{p}$.

When $x = 8$, p is true and q is false. Therefore, for this value of x, the conditional $(p \to q)$ is false and its converse $(q \to p)$ is true.

In this example, the conditional is false and the converse is true. Again, the conditional and its converse do not have the same truth value.

3. *A conditional and its converse can have the same truth value.*

Let p represent "Today is Friday."
Let q represent "Tomorrow is Saturday."

Conditional ($p \to q$): If $\underbrace{\text{today is Friday}}_{p}$, then $\underbrace{\text{tomorrow is Saturday}}_{q}$.

Converse ($q \to p$): If $\underbrace{\text{tomorrow is Saturday}}_{q}$, then $\underbrace{\text{today is Friday}}_{p}$.

On Friday, p is true and q is true. Therefore, both $(p \to q)$ and $(q \to p)$ are true.

On any other day of the week, p is false and q is false. Therefore, both $(p \to q)$ and $(q \to p)$ are true.

A conditional and its converse may have the same truth value.

These three examples allow us to make the following conclusion:

► **A conditional $(p \to q)$ and its converse $(q \to p)$ may or may not have the same truth value.**

This conclusion can be shown in the truth table. Note that the conditional $(p \to q)$ and its converse $(q \to p)$ have the same truth value when p and q have the same truth value. The conditional $(p \to q)$ and its converse $(q \to p)$ have different truth values when p and q have different truth values.

		Conditional	*Converse*
p	q	$p \to q$	$q \to p$
T	T	T	T
T	F	F	T
F	T	T	F
F	F	T	T

The Contrapositive

We form the inverse of a conditional by negating both the hypothesis and the conclusion. We form the converse of a conditional by interchanging the hypothesis and the conclusion. We form the **contrapositive** of a conditional by doing both of these operations: we negate and interchange the hypothesis and conclusion. In symbols, the contrapositive of $(p \to q)$ is $(\sim q \to \sim p)$.

1. *A true conditional can have a true contrapositive.*

Let p represent "Gary arrives late to class."
Let q represent "Gary is marked tardy."

Conditional $(p \to q)$: If $\underbrace{\text{Gary arrives late to class}}_{p}$, then $\underbrace{\text{Gary is marked tardy}}_{q}$.

Contrapositive $(\sim q \to \sim p)$: If $\underbrace{\text{Gary is } \textit{not} \text{ marked tardy}}_{\sim q}$,

then $\underbrace{\text{Gary does } \textit{not} \text{ arrive late to class}}_{\sim p}$.

If Gary arrives late to class, p is true and q is true. Therefore, $(p \to q)$ is true. Also, if p and q are true, $\sim p$ is false and $\sim q$ is false, so $(\sim q \to \sim p)$ is true.

If p is false, that is, if Gary does not arrive late to class, q is false and $(p \to q)$ is true. Similarly, if p and q are false, $\sim p$ is true and $\sim q$ is true, so $(\sim q \to \sim p)$ is true.

2. *A false conditional can have a false contrapositive.*

Conditional ($p \to q$): If $\underbrace{x \text{ is an odd number}}_{p}$, then $\underbrace{x \text{ is a prime number}}_{q}$.

Contrapositive ($\sim q \to \sim p$): If $\underbrace{x \text{ is not a prime number}}_{\sim q}$, then $\underbrace{x \text{ is not an odd number}}_{\sim p}$.

	$p \to q$	$\sim q \to \sim p$
Let $x = 2$	F $\to$ T is true	F $\to$ T is true
Let $x = 9$	T $\to$ F is false	T $\to$ F is false
Let $x = 11$	T $\to$ T is true	F $\to$ F is true
Let $x = 8$	F $\to$ F is true	T $\to$ T is true

For each value of x, the conditional and its contrapositive have the same truth value.

These illustrations allow us to make the following conclusion:

▶ **A conditional ($p \to q$) and its contrapositive ($\sim q \to \sim p$) always have the same truth value:**

When a conditional is true, its contrapositive must be true.

When a conditional is false, its contrapositive must be false.

This conclusion can be shown in the following truth table.

				Conditional	*Contrapositive*
p	q	$\sim p$	$\sim q$	$p \to q$	$\sim q \to \sim p$
T	T	F	F	T	T
T	F	F	T	F	F
F	T	T	F	T	T
F	F	T	T	T	T

Logical Equivalents

A conditional and its contrapositive are **logical equivalents** because they always have the same truth value.

The inverse of $(p \to q)$ is $(\sim p \to \sim q)$. The contrapositive of $(\sim p \to \sim q)$ is formed by negating and interchanging the hypothesis and conclusion of $(\sim p \to \sim q)$. The negation of $\sim p$ is p and the negation of $\sim q$ is q. Therefore, the contrapositive of $(\sim p \to \sim q)$ is $(q \to p)$. For instance:

Conditional ($p \to q$): If $\underbrace{\text{Jessica likes waffles}}_{p}$, then $\underbrace{\text{Jessica eats waffles}}_{q}$.

Inverse ($\sim p \to \sim q$): If $\underbrace{\text{Jessica does not like waffles}}_{\sim p}$,
then $\underbrace{\text{Jessica does not eat waffles}}_{\sim q}$.

Contrapositive of the inverse ($q \to p$): If $\underbrace{\text{Jessica eats waffles}}_{q}$, then $\underbrace{\text{Jessica likes waffles}}_{p}$.

Notice, however, that the contrapositive of the inverse is the same as the converse of the original conditional. Thus, the inverse and the converse of $(p \to q)$ are contrapositives of each other. Since a conditional and its contrapositive always have the same truth value, the converse and the inverse always have the same truth value. This can be verified by constructing the following truth table.

p	q	$\sim p$	$\sim q$	$\sim p \to \sim q$	$q \to p$
T	T	F	F	T	T
T	F	F	T	T	T
F	T	T	F	F	F
F	F	T	T	T	T

EXAMPLE 1

Write the inverse, converse, and contrapositive of the given conditional:
If today is Tuesday, then I play basketball.

Solution

Inverse: If today is not Tuesday, then I do not play basketball.
Converse: If I play basketball, then today is Tuesday.
Contrapositive: If I do not play basketball, then today is not Tuesday. ❑

EXAMPLE 2

Write the inverse, converse, and contrapositive of the given conditional:
If a polygon is a square then it has four right angles.

Solution

Inverse: If a polygon is not a square, then it does not have four right angles.
Converse: If a polygon has four right angles, then it is a square.
Contrapositive: If a polygon does not have four right angles, then it is not a square. ❑

EXAMPLE 3

Write the inverse, converse, and contrapositive of the given conditional:

If M is the midpoint of $\overline{AB}$, then $AM = MB$.

Solution

Inverse: If M is not the midpoint of $\overline{AB}$, then $AM \neq MB$.
Converse: If $AM = MB$, then M is the midpoint of $\overline{AB}$.
Contrapositive: If $AM \neq MB$, then M is not the midpoint of $\overline{AB}$.

EXAMPLE 4

Given the true statement, "If the polygon is a rectangle, then it has four sides," which statement must also be true?

(1) If the polygon has four sides, then it is a rectangle.

(2) If the polygon is not a rectangle, then it does not have four sides.

(3) If the polygon does not have four sides, then it is not a rectangle.

(4) If the polygon has four sides, then it is not a rectangle.

Solution A conditional and its contrapositive always have the same truth value. The contrapositive of the given statement is "If a polygon does not have four sides then it is not a rectangle."

Answer (3)

Exercises

Writing About Mathematics

1. Samuel said that if you know that a conditional is true then you know that the converse of the conditional is true. Do you agree with Samuel? Explain why or why not.

2. Kate said that if you know the truth value of a conditional and of its converse then you know the truth value of the inverse and the contrapositive. Do you agree with Kate? Explain why or why not.

Developing Skills

In 3–6, for each statement, write in symbolic form:

a. the inverse **b.** the converse **c.** the contrapositive.

3. $p \rightarrow q$ **4.** $t \rightarrow \sim w$ **5.** $\sim m \rightarrow p$ **6.** $\sim p \rightarrow \sim q$

In 7–10: **a.** Write the inverse of each conditional statement in words. **b.** Give the truth value of the conditional. **c.** Give the truth value of the inverse.

7. If $6 > 3$, then $-6 > -3$.

8. If a polygon is a parallelogram, then the polygon has two pairs of parallel sides.

9. If $3(3) = 9$, then $3(4) = 12$.

10. 1f $2^2 = 4$, then $3^2 = 6$.

In 11–14, write the converse of each statement in words.

11. If you lower your cholesterol, then you eat Quirky oatmeal.

12. If you enter the Grand Prize drawing, then you will get rich.

13. If you use Shiny's hair cream, then your hair will curl.

14. If you feed your pet Krazy Kibble, he will grow three inches.

In 15–18: **a.** Write the converse of each conditional statement in words. **b.** Give the truth value of the conditional. **c.** Give the truth value of the converse.

15. If a number is even, then the number is exactly divisible by 2.

16. If 0.75 is an integer, then it is rational.

17. If $8 = 1 + 7$, then $8^2 = 1^2 + 7^2$.

18. If $4(5) - 6 = 20 - 6$, then $4(5) - 6 = 14$.

In 19–23: **a.** Write the contrapositive of each statement in words. **b.** Give the truth value of the conditional. **c.** Give the truth value of the contrapositive.

19. If Rochester is a city, then Rochester is the capital of New York.

20. If two angles form a linear pair, then they are supplementary.

21. If $3 - 2 = 1$, then $4 - 3 = 2$.

22. If all angles of a triangle are equal in measure, then the triangle is equiangular.

23. If $\frac{1}{2} > 0$, then $\frac{1}{2}$ is a counting number.

In 24–28, write the numeral preceding the expression that best answers the question.

24. When $p \rightarrow q$ is true, which related conditional must be true?
(1) $q \rightarrow p$ (2) $\sim p \rightarrow \sim q$ (3) $p \rightarrow \sim q$ (4) $\sim q \rightarrow \sim p$

25. Which is the contrapositive of "If March comes in like a lion, it goes out like a lamb"?
(1) If March goes out like a lamb, then it comes in like a lion.
(2) If March does not go out like a lamb, then it comes in like a lion.
(3) If March does not go out like a lamb, then it does not come in like a lion.
(4) March goes out like a lion if it comes in like a lamb.

26. Which is the converse of "If a rectangular prism is a cube, then its surface area is $6s^2$"?
(1) If a rectangular prism is not a cube, then its surface area is not $6s^2$.
(2) If the surface area of a rectangular prism is $6s^2$, then it is a cube.
(3) If the surface area of a rectangular prism is not $6s^2$, then it is not a cube.
(4) If the surface area of a cube is $6s^2$, then it is a rectangular prism.

27. Which is the inverse of "If $z = 4$, then $2z \neq 9$"?
(1) If $2z \neq 9$, then $z = 4$.
(2) If $2z = 9$, then $z \neq 4$.
(3) If $z \neq 4$, then $2z = 9$.
(4) If $z \neq 4$, then $2z \neq 9$.

28. Which is the contrapositive of "If y is greater than 3, then $2y + 10y$ is not equal to 36"?
(1) If $2y + 10y$ is not equal to 36, then y is greater than 3.
(2) If $2y + 10y$ equals 36, then y is not greater than 3.
(3) If $2y + 10y$ is not equal to 36, then y is not greater than 3.
(4) If $2y + 10y$ equals 36, then y is greater than 3.

Applying Skills

In 29–34, assume that each conditional statement is true. Then:

a. Write its converse in words and state whether the converse is always true, sometimes true, or never true.

b. Write its inverse in words and state whether the inverse is always true, sometimes true, or never true.

c. Write its contrapositive in words and state whether the contrapositive is always true, sometimes true, or never true.

29. If Derek lives in Las Vegas, then he lives in Nevada.

30. If a bin contains 3 red marbles and 3 blue marbles, then the probability of picking a red marble from the bin is $\frac{1}{2}$.

31. If a polygon has eight sides, then it is an octagon.

32. If a garden grows carrots, then it grows vegetables.

33. If the dimensions of a rectangle are 8 feet by 6 feet, then the area of the rectangle is 48 square feet.

34. If a number has 7 as a factor, then it is divisible by 7.

2-6 BICONDITIONALS

A **biconditional** is the conjunction of a conditional and its converse. For the conditional $(p \rightarrow q)$, the converse is $(q \rightarrow p)$. The biconditional can be written as $(p \rightarrow q) \wedge (q \rightarrow p)$ or in the shorter form $p \leftrightarrow q$, which is read *p if and only if q*.

Recall that a conjunction is true only when both parts of the compound statement are true. Therefore, $(p \to q) \wedge (q \to p)$ is true only when $(p \to q)$ is true and its converse $(q \to p)$ is true. In the last section you learned that a conditional $(p \to q)$ and its converse $(q \to p)$ are both true when p and q are both true or both false. This is shown in the table below.

p	q	$p \to q$	$q \to p$	$(p \to q) \wedge (q \to p)$	$p \leftrightarrow q$
T	T	T	T	T	T
T	F	F	T	F	F
F	T	T	F	F	F
F	F	T	T	T	T

► **The biconditional p *if and only if* q is true when p and q are both true or both false.**

In other words, $p \leftrightarrow q$ is true when p and q have the same truth value. When p and q have different truth values, the biconditional is false.

Applications of the Biconditional

There are many examples in which the biconditional is always true. Consider the following:

1. *Every definition is a true biconditional.*

Every definition can be written in reverse order. Both of the following statements are true:

- Congruent segments are segments that have the same measure.
- Line segments that have the same measure are congruent.

We can restate the definition as two true conditionals:

- If two line segments are congruent, then they have the same measure.
- If two line segments have the same measure, then they are congruent.

Therefore, this definition can be restated as a true biconditional:

- Two line segments are congruent if and only if they have the same measure.

2. *Biconditionals are used to solve equations.*

We know that when we add the same number to both sides of an equation or when we multiply both sides of an equation by the same number, the derived equation has the same solution set as the given equation. That is, any number that makes the first equation true will make the derived equation true.

For example:

p: $3x + 7 = 19$
q: $3x = 12$
$p \to q$: If $3x + 7 = 19$, then $3x = 12$. (-7 was added to both sides of the equation.)
$q \to p$: If $3x = 12$, then $3x + 7 = 19$. (7 was added to both sides of the equation.)

When $x = 4$, both p and q are true and both $p \to q$ and $q \to p$ are true.

When $x = 1$ or when x equals any number other than 3, both p and q are false and both $p \to q$ and $q \to p$ are true.

Therefore, the biconditional "$3x + 7 = 19$ if and only if $3x = 12$" is true.

The solution of an equation is a series of biconditionals:

$$\begin{aligned} 3x + 7 &= 19 \\ 3x &= 12 && 3x + 7 = 19 \text{ if and only if } 3x = 12. \\ x &= 4 && 3x = 12 \text{ if and only if } x = 4. \end{aligned}$$

3. *A biconditional states that two logical forms are equivalent.*

Two logical forms that always have the same truth values are said to be equivalent.

We have seen that a conditional and its contrapositive are logically equivalent and that the converse and inverse of a conditional are logically equivalent. There are many other statements that are logically equivalent.

The table below shows that $\sim(p \wedge q)$ and $\sim p \vee \sim q$ are logically equivalent. We can write the true biconditional $\sim(p \wedge q) \leftrightarrow (\sim p \vee \sim q)$.

p	q	$\sim p$	$\sim q$	$p \wedge q$	$\sim(p \wedge q)$	$\sim p \vee \sim q$
T	T	F	F	T	F	F
T	F	F	T	F	T	T
F	T	T	F	F	T	T
F	F	T	T	F	T	T

EXAMPLE 1

Determine the truth value to be assigned to the biconditional.

Germany is a country in Europe if and only if Berlin is the capital of Germany.

Solution "Germany is a country in Europe" is true.
"Berlin is the capital of Germany" is true.

Therefore, the biconditional "Germany is a country in Europe if and only if Berlin is the capital of Germany" is also true. *Answer*

EXAMPLE 2

The statement "I go to basketball practice on Monday and Thursday" is true. Determine the truth value to be assigned to each statement.

a. If today is Monday, then I go to basketball practice.

b. If I go to basketball practice, then today is Monday.

c. Today is Monday if and only if I go to basketball practice.

Solution Let p: "Today is Monday,"
and q: "I go to basketball practice."

a. We are asked to find the truth value of the following conditional:

$p \to q$: If today is Monday, then I go to basketball practice.

On Monday, p is true and q is true. Therefore, $p \to q$ is true.

On Thursday, p is false and q is true. Therefore, $p \to q$ is true.

On every other day, p is false and q is false. Therefore, $p \to q$ is true.

"If today is Monday, then I go to basketball practice" is always true. *Answer*

b. We are asked to find the truth value of the following conditional:

$q \to p$: If I go to basketball practice then today is Monday.

On Monday, p is true and q is true. Therefore, $q \to p$ is true.

On Thursday, p is false and q is true. Therefore, $q \to p$ is false.

On every other day, p is false and q is false. Therefore, $q \to p$ is true.

"If I go to basketball practice, then Today is Monday" is sometimes true and sometimes false. *Answer*

c. We are asked to find the truth value of the following biconditional:

$p \leftrightarrow q$: Today is Monday if and only if I go to basketball practice.

The conditionals $p \to q$ and $q \to p$ do not always have the same truth value. Therefore, the biconditional "Today is Monday if and only if I go to basketball practice" is not always true. We usually say that a statement that is not always true is false. *Answer* ∎

EXAMPLE 3

Determine the truth value of the biconditional.

$$3y + 1 = 28 \text{ if and only if } y = 9.$$

Solution When $y = 9$, $3y + 1 = 28$ is true and $y = 9$ is true.
Therefore, $3y + 1 = 28$ if and only if $y = 9$ is true.
When $y \neq 9$, $3y + 1 = 28$ is false and $y = 9$ is false.
Therefore, $3y + 1 = 28$ if and only if $y = 9$ is true.
$3y + 1 = 28$ if and only if $y = 9$ is always true. *Answer*

Exercises

Writing About Mathematics

1. Write the definition "A prime number is a whole number greater than 1 and has exactly two factors" as a biconditional.

2. Tiffany said that if the biconditional $p \leftrightarrow q$ is false, then either $p \rightarrow q$ is true or $q \rightarrow p$ is true but both cannot be true. Do you agree with Tiffany? Explain why or why not.

Developing Skills

In 3–16, give the truth value of each biconditional.

3. $y + 7 = 30$ if and only if $y = 23$.

4. B is between A and C if and only if $AB + AC = BC$.

5. $z + 9 = 13$ if and only if $z + 2 = 6$.

6. A parallelogram is a rhombus if and only if the parallelogram has four sides of equal length.

7. A real number is positive if and only if it is greater than zero.

8. An angle is an acute angle if and only if its degree measure is less than 90.

9. An element belongs to the intersection of sets F and G if and only if it belongs to both F and G.

10. An integer is odd if and only if it is not divisible by 2.

11. Two angles have the same measure if and only if they are right angles.

12. I live in the United States if and only if I live in New York State.

13. A rational number has a multiplicative inverse if and only if it is not zero.

14. An angle is an acute angle if and only if it has a degree measure of 50.

15. $x = 5$ if and only if $x > 3$.

16. Today is Friday if and only if tomorrow is Saturday.

Applying Skills

17. Let p represent "x is divisible by 2."
Let q represent "x is divisible by 3."
Let r represent "x is divisible by 6."

a. Write the biconditional $(p \wedge q) \leftrightarrow r$ in words.

b. Show that the biconditional is always true for the domain $\{2, 3, 5, 6, 8, 9, 11, 12\}$.

c. Do you think that the biconditional is true for all counting numbers? Explain your answer.

18. A gasoline station displays a sign that reads "Open 24 hours a day, Monday through Friday."

a. On the basis of the information on the sign, is the conditional "The gasoline station is closed if it is Saturday or Sunday" true?

b. On the basis of the information on the sign, is the conditional "If the gasoline station is closed, it is Saturday or Sunday" true?

c. On the basis of the information on the sign, is the biconditional "The gasoline station is closed if and only if it is Saturday or Sunday" true?

d. Marsha arrives at the gasoline station on Monday and finds the station closed. Does this contradict the information on the sign?

e. Marsha arrives at the gasoline station on Saturday and finds the station open. Does this contradict the information on the sign?

In 19–22, write a biconditional using the given conditionals and tell whether each biconditional is true or false.

19. If a triangle is isosceles, then it has two congruent sides.
If a triangle has two congruent sides then it is isosceles.

20. If two angles are both right angles, then they are congruent.
If two angles are congruent, then they are both right angles.

21. If today is Thursday, then tomorrow is not Saturday.
If tomorrow is not Saturday, then today is Thursday.

22. If today is not Friday, then tomorrow is not Saturday.
If tomorrow is not Saturday, then today is not Friday.

2-7 THE LAWS OF LOGIC

We frequently want to combine known facts in order to establish the truth of related facts. To do this, we can look for patterns that are frequently used in drawing conclusions. These patterns are called the **laws of logic**.

The Law of Detachment

A **valid argument** uses a series of statements called **premises** that have known truth values to arrive at a conclusion.

For example, Cynthia makes the following true statements to her parents:

I want to play baseball.
If I want to play baseball, then I need a glove.

These are the premises of Cynthia's argument. The conclusion that Cynthia wants her parents to make is that she needs a glove. Is this conclusion valid?

Let p represent "I want to play baseball."
Let q represent "I need a glove."
Then $p \to q$ represents "If I want to play baseball, then I need a glove."

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

We know that the premises are true, that is, p is true and $p \to q$ is true. The only line of the truth table that satisfies both of these conditions is the first in which q is also true. Therefore, "I need a glove" is a true conclusion.

The example just given does not depend on the statement represented by p and q. The first line of the truth table tells us that whenever $p \to q$ is true and p is true, then q must be a true conclusion. This logical pattern is called the **Law of Detachment**:

► **If a conditional ($p \to q$) is true and the hypothesis (p) is true, then the conclusion (q) is true.**

EXAMPLE 1

If the measure of an angle is greater than 0° and less than 90°, then the angle is an acute angle. Let "m$\angle A$ = 40°, which is greater than 0° and less than 90°" be a true statement. Prove that $\angle A$ is an acute angle.

Solution Let p represent "m$\angle A$ = 40°, which is greater than 0° and less than 90°."

Let q represent "the angle is an acute angle."

Then $p \to q$ is true because it is a definition of an acute angle.

Also, p is true because it is given.

Then by the Law of Detachment, q is true.

Answer "$\angle A$ is an acute angle" is true. ❑

The Law of Disjunctive Inference

We know that a disjunction is true when one or both statements that make up the disjunction are true. The disjunction is false when both statements that make up the disjunction are false.

For example, let p represent "A real number is rational" and q represent "A real number is irrational." Then $p \vee q$ represents "A real number is rational or a real number is irrational," a true statement.

When the real number is π, then "A real number is rational" is false. Therefore "A real number is irrational" must be true.

When the real number is 7, then "A real number is irrational" is false. Therefore "A real number is rational" must be true.

The truth table shows us that when p is false and $p \vee q$ is true, only the third line of the table is satisfied. This line tells us that q is true. Also, when q is false and $p \vee q$ is true, only the second line of the table is satisfied. This line tells us that p is true.

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

The example just given illustrates a logical pattern that does not depend on the statements represented by p and q. When a disjunction is true and one of the disjuncts is false, then the other disjunct must be true. This logical pattern is called the **Law of Disjunctive Inference**:

- **If a disjunction ($p \vee q$) is true and the disjunct (p) is false, then the other disjunct (q) is true.**
- **If a disjunction ($p \vee q$) is true and the disjunct (q) is false, then the other disjunct (p) is true.**

EXAMPLE 2

What conclusion can be drawn when the following statements are true?

I will walk to school or I will ride to school with my friend.
I do not walk to school.

Solution Since "I do not walk to school" is true, "I walk to school" is false. The disjunction "I will walk to school or I will ride to school with my friend" is true and one of the disjuncts, "I walk to school," is false. By the Law of Disjunctive Inference, the other disjunct, "I ride to school with my friend," must be true.

Alternative Solution Let p represent "I walk to school."
Let q represent "I ride to school with my friend."

Make a truth table for the disjunction $p \vee q$ and eliminate the rows that do not apply.

We know that $p \vee q$ is true. We also know that since $\sim p$ is true, p is false.

(1) Eliminate the last row of truth values in which the disjunction is false.

(2) Eliminate the first two rows of truth values in which p is true.

(3) Only one case remains: q is true.

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Answer I ride to school with my friend.

Note: In most cases, more than one possible statement can be shown to be true. For example, the following are also true statements when the given statements are true:

I do not walk to school and I ride to school with my friend.
If I do not walk to school, then I ride to school with my friend.
If I do not ride to school with my friend, then I walk to school.

EXAMPLE 3

From the following true statements, is it possible to determine the truth value of the statement "I will go to the library"?

If I have not finished my essay for English class, then I will go to the library.
I have finished my essay for English class.

Solution When "I have finished my essay for English class" is true, its negation, "I have not finished my essay for English class" is false. Therefore, the true conditional "If I have not finished my essay for English class, then I will go to the library" has a false hypothesis and the conclusion, "I will go to the library" can be either true or false.

Alternative Solution Let p represent "I have not finished my essay for English class."
Let q represent "I will go to the library."

Make a truth table for the disjunction $p \rightarrow q$ and eliminate the rows that do not apply.

We know that $p \rightarrow q$ is true. We also know that since $\sim p$ is true, p is false.

(1) Eliminate the second row of truth values in which the conditional is false.

(2) Eliminate the first row of truth values in which p is true. The second row in which p is true has already been eliminated.

(3) Two rows remain, one in which q is true and the other in which q is false.

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

Answer "I will go to the library" could be either true or false.

EXAMPLE 4

Draw a conclusion or conclusions base on the following true statements.

If I am tired, then I will rest.
I do not rest.

Solution A conditional and its contrapositive are logically equivalent. Therefore, "If I do not rest, then I am not tired" is true.

By the Law of Detachment, when the hypothesis of a true conditional is true, the conclusion must be true. Therefore, since "I do not rest" is true, "I am not tired" must be true.

Alternative Solution Let p represent "I am tired."
Let q represent "I will rest."

Make a truth table for the disjunction $p \to q$ and eliminate the rows that do not apply.

We know that $p \to q$ is true. We also know that since $\sim q$ is true, q is false.

(1) Eliminate the second row of truth values in which the conditional is false.

(2) Eliminate the first and third rows of truth values in which q is true.

(3) Only one case remains: p is false.

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

Since p is false, $\sim p$ is true. "I am not tired" is true.

Answer I am not tired.

Exercises

Writing About Mathematics

1. Clovis said that when $p \to q$ is false and $q \vee r$ is true, r must be true. Do you agree with Clovis? Explain why or why not.

2. Regina said when $p \vee q$ is true and $\sim q$ is true, then $p \wedge \sim q$ must be true. Do you agree with Regina? Explain why or why not.

Developing Skills

In 3–14, assume that the first two sentences in each group are true. Determine whether the third sentence is true, is false, or cannot be found to be true or false. Justify your answer.

3. I save up money or I do not go on the trip.
I go on the trip.
I save up money.

4. If I speed, then I get a ticket.
I speed.
I get a ticket.

5. I like swimming or kayaking.
I like kayaking.
I like swimming.

6. I like swimming or kayaking.
I do not like swimming.
I like kayaking.

7. $x \leq 18$ if $x = 14$.
$x = 14$
$x \leq 18$

8. I live in Pennsylvania if I live in Philadelphia.
I do not live in Philadelphia.
I live in Pennsylvania.

9. If I am late for dinner, then my dinner will be cold.
I am late for dinner.
My dinner is cold.

10. If I am late for dinner, then my dinner will be cold.
I am not late for dinner.
My dinner is not cold.

11. I will go to college if and only if I work this summer.
I do not work this summer.
I will go to college.

12. The average of two numbers is 20 if the numbers are 17 and 23.
The average of two numbers is 20.
The two numbers are 17 and 23.

13. If I am late for dinner, then my dinner will be cold.
My dinner is not cold.
I am not late for dinner.

14. If I do not do well in school, then I will not receive a good report card.
I do well in school.
I receive a good report card.

Applying Skills

In 15–27, assume that each given sentence is true. Write a conclusion using both premises, if possible. If no conclusion is possible, write "No conclusion." Justify your answer.

15. If I play the trumpet, I take band.
I play the trumpet.

16. $\sqrt{6}$ is rational or irrational.
$\sqrt{6}$ is not rational.

17. If $2b + 6 = 14$, then $2b = 8$.
If $2b = 8$, then $b = 4$.
$2b + 6 = 14$

18. If it is 8:15 A.M., then it is morning.
It is not morning.

19. If k is a prime, then $k \neq 8$.
$k = 8$

20. x is even and a prime if and only if $x = 2$.
$x = 2$

21. It is February or March, and it is not summer.
It is not March.

22. If x is divisible by 4, then x is divisible by 2.
x is divisible by 2.

23. On Saturdays, we go bowling or we fly kites.
Last Saturday, we did not go bowling.

24. I study computer science, and wood shop or welding.
I do not take woodshop.

25. Five is a prime if and only if five has exactly two factors.
Five is a prime.

26. If x is an integer greater than 2 and x is a prime, then x is odd.
x is not an odd integer.

27. If a ray bisects an angle, the ray divides the angle into two congruent angles.
Ray DF does not divide angle CDE into two congruent angles.

2-8 DRAWING CONCLUSIONS

Many important decisions as well as everyday choices are made by applying the principles of logic. Games and riddles also often depend on logic for their solution.

EXAMPLE 1

The three statements given below are each true. What conclusion can be found to be true?

1. If Rachel joins the choir then Rachel likes to sing.
2. Rachel will join the choir or Rachel will play basketball.
3. Rachel does not like to sing.

Solution A conditional and its contrapositive are logically equivalent. Therefore, "If Rachel does not like to sing, then Rachel will not join the choir" is true.

By the Law of Detachment, when the hypothesis of a true conditional is true, the conclusion must be true. Therefore, since "Rachel does not like to sing" is true, "Rachel will not join the choir" must be true.

Since "Rachel will not join the choir" is true, then its negation, "Rachel will join the choir," must be false.

Since "Rachel will join the choir or Rachel will play basketball" is true and "Rachel will join the choir" is false, then by the Law of Disjunctive Inference, "Rachel will play basketball" must be true.

Answer Rachel does not join the choir. Rachel will play basketball.

Alternative Solution Let c represent "Rachel joins the choir,"
s represent "Rachel likes to sing,"
and b represent "Rachel will play basketball."

Write statements 1, 2, and 3 in symbols:

1. $c \rightarrow s$ **2.** $c \vee b$ **3.** $\sim s$

Using Statement 1

$c \rightarrow s$ is true, so $\sim s \rightarrow \sim c$ is true. (A conditional and its contrapositive always have the same truth value.)

Using Statement 3

$\sim s$ is true and $\sim s \rightarrow \sim c$ is true, so $\sim c$ is true. (Law of Detachment)

Also, c is false.

Using Statement 2

$c \vee b$ is true and c is false, so b must be true. (Law of Disjunctive Inference)

Answer Rachel does not join the choir. Rachel will play basketball.

EXAMPLE 2

If Alice goes through the looking glass, then she will see Tweedledee. If Alice sees Tweedledee, then she will see the Cheshire Cat. Alice does not see the Cheshire Cat.

Show that Alice does *not* go through the looking glass.

Solution A conditional and its contrapositive are logically equivalent. Therefore, "If Alice does not see the Cheshire Cat, then Alice does not see Tweedledee" is true.

By the Law of Detachment, when the hypothesis of a true conditional is true, the conclusion must be true. Therefore, since "Alice does not see the Cheshire Cat" is true, "Alice does not see Tweedledee" must be true.

Again, since a conditional and its contrapositive are logically equivalent, "If Alice does not see Tweedledee, then Alice does not go through the looking

glass" is also true. Applying the Law of Detachment, since "Alice does not see Tweedledee" is true, "Alice does not go through the looking glass" must be true.

Alternative Solution Let a represent "Alice goes through the looking glass."

t represent "Alice sees Tweedledee."

c represent "Alice sees the Cheshire cat."

Then, in symbols, the given statements are:

1. $a \to t$ **2.** $t \to c$ **3.** $\sim c$

We would like to conclude that $\sim a$ is true.

Using Statement 2

$t \to c$ is true, so $\sim c \to \sim t$ is true. (A conditional and its contrapositive always have the same truth value.)

Using Statement 3

$\sim c$ is true and $\sim c \to \sim t$ is true, so $\sim t$ is true. (Law of Detachment)

Using Statement 1

$a \to t$ is true, so $\sim t \to \sim a$ is true. (A conditional and its contrapositive always have the same truth value.)

Therefore, by the Law of Detachment, since $\sim t$ is true and $\sim t \to \sim a$ is true, $\sim a$ is true. ∎

EXAMPLE 3

Three siblings, Ted, Bill, and Mary, each take a different course in one of three areas for their senior year: mathematics, art, and thermodynamics. The following statements about the siblings are known to be true.

- Ted tutors his sibling taking the mathematics course.
- The art student and Ted have an argument over last night's basketball game.
- Mary loves the drawing made by her sibling taking the art course.

What course is each sibling taking?

Solution Make a table listing each sibling and each subject. Use the statements to fill in the information about each sibling.

(1) Since Ted tutors his sibling taking the math course, Ted cannot be taking math. Place an ✗ in the table to show this.

	Math	Art	Thermodynamics
Ted	✗		
Bill			
Mary			

(2) Similarly, the second statement shows that Ted cannot be taking art. Place an ✗ to indicate this. The only possibility left is for Ted to be taking thermodynamics. Place a ✓ to indicate this, and add ✗'s in the thermodynamics column to show that no one else is taking this course.

	Math	Art	Thermodynamics
Ted	✗	✗	✓
Bill			✗
Mary			✗

(3) The third statement shows that Mary is not taking art. Therefore, Mary is taking math. Since Ted is taking thermodynamics and Mary is taking math, Bill must be taking art.

	Math	Art	Thermodynamics
Ted	✗	✗	✓
Bill	✗	✓	✗
Mary	✓	✗	✗

Answer Ted takes thermodynamics, Bill takes art, and Mary takes mathematics. ❑

Exercises

Writing About Mathematics

1. Can $(p \vee q) \wedge r$ be true when p is false? If so, what are the truth values of q and of r? Justify your answer.

2. Can $(p \vee q) \wedge r$ be true when r is false? If so, what are the truth values of p and of q? Justify your answer.

Developing Skills

3. When $\sim p \wedge \sim q$ is true and $q \vee r$ is true, what is the truth value of r?

4. When $p \rightarrow q$ is false and $q \vee r$ is true, what is the truth value of r?

5. When $p \rightarrow q$ is false, what is the truth value of $q \rightarrow r$?

6. When $p \rightarrow q$ and $p \wedge q$ are both true, what are the truth values of p and of q?

7. When $p \rightarrow q$ and $p \wedge q$ are both false, what are the truth values of p and of q?
8. When $p \rightarrow q$ is true and $p \wedge q$ is false, what are the truth values of p and of q?
9. When $p \rightarrow q$ is true, $p \vee r$ is true and q is false, what is the truth value of r?

Applying Skills

10. Laura, Marta, and Shanti are a lawyer, a doctor, and an investment manager.
    - The lawyer is Marta's sister.
    - Laura is not a doctor.
    - Either Marta or Shanti is a lawyer.

    What is the profession of each woman?
11. Alex, Tony, and Kevin each have a different job: a plumber, a bookkeeper, and a teacher.
    - Alex is a plumber or a bookkeeper.
    - Tony is a bookkeeper or a teacher.
    - Kevin is a teacher.

    What is the profession of each person?
12. Victoria owns stock in three companies: Alpha, Beta, and Gamma.
    - Yesterday, Victoria sold her shares of Alpha or Gamma.
    - If she sold Alpha, then she bought more shares of Beta.
    - Victoria did not buy more shares of Beta.

    Which stock did Victoria sell yesterday?
13. Ren, Logan, and Kadoogan each had a different lunch. The possible lunches are: a ham sandwich, pizza, and chicken pot pie.
    - Ren or Logan had chicken pot pie.
    - Kadoogan did not have pizza.
    - If Logan did not have pizza, then Kadoogan had pizza.

    Which lunch did each person have?
14. Zach, Steve, and David each play a different sport: basketball, soccer, or baseball. Zach made each of the following true statements.
    - I do not play basketball.
    - If Steve does not play soccer, then David plays baseball.
    - David does not play baseball.

    What sport does each person play?

15. Taylor, Melissa, and Lauren each study one language: French, Spanish, and Latin.

- If Melissa does not study French, then Lauren studies Latin.
- If Lauren studies Latin, then Taylor studies Spanish.
- Taylor does not study Spanish.

What language does each person study?

16. Three friends, Augustus, Brutus, and Caesar, play a game in which each decides to be either a liar or a truthteller. A liar must always lie and a truthteller must always tell the truth. When you met these friends, you asked Augustus which he had chosen to be. You didn't hear his answer but Brutus volunteered, "Augustus said that he is a liar." Caesar added, "If one of us is a liar, then we are all liars." Can you determine, for each person, whether he is a liar or a truthteller?

CHAPTER SUMMARY

Definitions to Know

- **Logic** is the study of reasoning.
- In logic, a **mathematical sentence** is a sentence that contains a complete thought and can be judged to be true or false.
- A **phrase** is an expression that is only part of a sentence.
- An **open sentence** is any sentence that contains a variable.
- The **domain** or **replacement set** is the set of numbers that can replace a variable.
- The **solution set** or **truth set** is the set of all replacements that will change an open sentence to true sentences.
- A **statement** or a **closed sentence** is a sentence that can be judged to be true or false.
- A closed sentence is said to have a **truth value**, either true **(T)** or false **(F)**.
- The **negation** of a statement has the opposite truth value of a given statement.
- In logic, a **compound sentence** is a combination of two or more mathematical sentences formed by using the connectives *not*, *and*, *or*, *if . . . then*, or *if and only if*.
- A **conjunction** is a compound statement formed by combining two simple statements, called **conjuncts**, with the word *and*. The conjunction ***p* and *q*** is written symbolically as $p \wedge q$.
- A **disjunction** is a compound statement formed by combining two simple statements, called **disjuncts**, with *or*. The disjunction ***p* or *q*** is written symbolically as $p \vee q$.
- A **truth table** is a summary of all possible truth values of a logic statement.
- A **conditional** is a compound statement formed by using the words *if . . . then* to combine two simple statements. The conditional **if *p* then *q*** is written symbolically as $p \rightarrow q$.

- A **hypothesis**, also called a **premise** or **antecedent**, is an assertion that begins an argument. The hypothesis usually follows the word *if*.
- A **conclusion**, also called a **consequent**, is an ending or a sentence that closes an argument. The conclusion usually follows the word *then*.
- Beginning with a statement $(p \to q)$, the **inverse $(\sim p \to \sim q)$** is formed by negating the hypothesis and negating the conclusion.
- Beginning with a statement $(p \to q)$, the **converse $(q \to p)$** is formed by interchanging the hypothesis and the conclusion.
- Beginning with a conditional $(p \to q)$, the **contrapositive $(\sim q \to \sim p)$** is formed by negating both the hypothesis and the conclusion, and then interchanging the resulting negation.
- Two statements are **logically equivalent**—or **logical equivalents**—if they always have the same truth value.
- A **biconditional $(p \leftrightarrow q)$** is a compound statement formed by the conjunction of the conditional $p \to q$ and its converse $q \to p$.
- A **valid argument** uses a series of statements called **premises** that have known truth values to arrive at a conclusion.

Logic Statements

Negation:	$\sim p$	not p
Conjunction:	$p \wedge q$	p and q
Disjunction:	$p \vee q$	p or q
Conditional:	$p \to q$	if p then q
Inverse:	$\sim p \to \sim q$	if $\sim p$ then $\sim q$
Converse:	$q \to p$	if q then p
Contrapositive:	$\sim q \to \sim p$	if $\sim q$ then $\sim p$
Biconditional:	$p \leftrightarrow q$	p if and only if q

The truth values of the logic connectives can be summarized as follows:

		Negation		Conjunction	Disjunction	Conditional	Inverse	Converse	Contrapositive	Biconditional
p	q	$\sim p$	$\sim q$	$p \wedge q$	$p \vee q$	$p \to q$	$\sim p \to \sim q$	$q \to p$	$\sim q \to \sim p$	$p \leftrightarrow q$
T	T	F	F	T	T	T	T	T	T	T
T	F	F	T	F	T	F	T	T	F	F
F	T	T	F	F	T	T	F	F	T	F
F	F	T	T	F	F	T	T	T	T	T

Laws of Logic

- The **Law of Detachment** states that when $p \rightarrow q$ is true and p is true, then q must be true.
- The **Law of Disjunctive Inference** states that when $p \vee q$ is true and p is false, then q must be true.

VOCABULARY

2-1 Logic • Truth value • Mathematical sentence • Phrase • Open sentence • Domain • Replacement Set • Solution set • Truth set • Statement • Closed sentence • Truth value (T and F) • Negation

2-2 Compound sentence • Compound statement • Conjunction • Conjunct • p and q • Tree diagram • Truth table

2-3 Disjunction • Disjunct • p or q • Inclusive *or* • Exclusive *or*

2-4 Conditional • If p then q • Hypothesis • Premise • Antecedent • Conclusion • Consequent

2-5 Inverse • Converse • Contrapositive • Logical equivalents

2-6 Biconditional

2-7 Laws of logic • Valid argument • Premises • Law of Detachment • Law of Disjunctive Inference

REVIEW EXERCISES

1. The statement "If I go to school, then I do not play basketball" is false. Using one or both of the statements "I go to school" and "I do not play basketball" or their negations, write five *true* statements.

2. The statement "If I go to school, then I do not play basketball" is false. Using one or both of the statements "I go to school" and "I do not play basketball" or their negations, write five *false* statements.

3. Mia said that the biconditional $p \leftrightarrow q$ and the biconditional $\sim p \leftrightarrow \sim q$ always have the opposite truth values. Do you agree with Mia? Explain why or why not.

In 4 and 5: **a**. Identify the hypothesis p. **b.** Identify the conclusion q.

4. If at first you don't succeed, then you should try again.

5. You will get a detention if you are late one more time.

In 6–12, tell whether each given statement is true or false.

6. If July follows June, then August follows July.

7. July follows June and July is a winter month in the northern hemisphere.

8. July is a winter month in the northern hemisphere or July follows June.

9. If August follows July, then July does not follow June.

10. July is a winter month if August is a winter month.

11. August does not follow July and July is not a winter month.

12. July follows June if and only if August follows July.

13. Which whole number, when substituted for y, will make the following sentence true?

$$(y + 5 > 9) \wedge (y < 6)$$

In 14–17, supply the word, phrase, or symbol that can be placed in each blank to make the resulting statement true.

14. $\sim(\sim p)$ has the same truth value as _____.

15. When p is true and q is false, then $p \wedge \sim q$ is _____.

16. When $p \vee \sim q$ is false, then p is _____ and q is _____.

17. If the conclusion q is true, then $p \rightarrow q$ must be _____.

In 18–22, find the truth value of each sentence when a, b, and c are all true.

18. $\sim a$ **19.** $\sim b \wedge c$ **20.** $b \rightarrow \sim c$ **21.** $a \vee \sim b$ **22.** $\sim a \leftrightarrow \sim b$

In 23–32, let p represent "$x > 5$," and let q represent "x is prime." Use the domain $\{1, 2, 3, 4, \ldots, 10\}$ to find the solution set for each of the following.

23. p **24.** $\sim p$ **25.** q **26.** $\sim q$ **27.** $p \vee q$

28. $p \wedge q$ **29.** $\sim p \wedge q$ **30.** $p \rightarrow \sim q$ **31.** $p \rightarrow q$ **32.** $p \leftrightarrow q$

33. For the conditional "If I live in Oregon, then I live in the Northwest," write: **a.** the inverse, **b.** the converse, **c.** the contrapositive, **d.** the biconditional.

34. Assume that the given sentences are true. Write a simple sentence that could be a conclusion.

- If $\angle A$ is the vertex angle of isosceles $\triangle ABC$, then $AB = AC$.
- $AB \neq AC$

35. Elmer Megabucks does not believe that girls should marry before the age of 21, and he disapproves of smoking. Therefore, he put the following provision in his will: I leave \$100,000 to each of my nieces who, at the time of my death, is over 21 or unmarried, and does not smoke.

Each of his nieces is described below at the time of Elmer's death. Which nieces will inherit \$100,000?

- Judy is 24, married, and smokes.
- Diane is 20, married, and does not smoke.

- Janice is 26, unmarried, and does not smoke.
- Peg is 19, unmarried, and smokes.
- Sue is 30, unmarried, and smokes.
- Sarah is 18, unmarried, and does not smoke.
- Laurie is 28, married, and does not smoke.
- Pam is 19, married, and smokes.

36. Some years after Elmer Megabucks prepared his will, he amended the conditions, by moving a comma, to read: I leave $100,000 to each of my nieces who, at the time of my death, is over 21, or unmarried and does not smoke. Which nieces described in Exercise 35 will now inherit $100,000?

37. At a swim meet, Janice, Kay, and Virginia were the first three finishers of a 200-meter backstroke competition. Virginia did not come in second. Kay did not come in third. Virginia came in ahead of Janice. In what order did they finish the competition?

38. Peter, Carlos, and Ralph play different musical instruments and different sports. The instruments that the boys play are violin, cello, and flute. The sports that the boys play are baseball, tennis, and soccer. From the clues given below, determine what instrument and what sport each boy plays.

- The violinist plays tennis.
- Peter does not play the cello.
- The boy who plays the flute does not play soccer.
- Ralph plays baseball.

39. Let p represent "x is divisible by 6."

Let q represent "x is divisible by 2."

a. If possible, find a value of x that will:

(1) make p true and q true. (2) make p true and q false.

(3) make p false and q true. (4) make p false and q false.

b. What conclusion can be drawn about the truth value of $p \rightarrow q$?

40. Each of the following statements is true.

- Either Peter, Jim, or Tom is Maria's brother.
- If Jim is Maria's brother, then Peter is Alice's brother.
- Alice has no brothers.
- Tom has no sisters.

Who is Maria's brother?

Exploration

1. A **tautology** is a statement that is always true. For instance, the disjunction $p \vee \sim p$ is a tautology because if p is true, the disjunction is true, and if p is false, $\sim p$ is true and the disjunction is true.

   a. Which of the following statements are tautologies?

   (1) $p \rightarrow (p \vee q)$

   (2) Either it will rain or it will snow.

   (3) $(p \wedge q) \rightarrow p$

   b. Construct two tautologies, one using symbols and the other using words.

2. A **contradiction** is a statement that is always false, that is, it cannot be true under any circumstances. For instance, the conjunction $p \wedge \sim p$ is a contradiction because if p is true, $\sim p$ is false and the conjunction is false, and if p is false, the conjunction is false.

   a. Which of the following statements are contradictions?

   (1) $(p \vee q) \wedge \sim p$

   (2) It is March or it is Tuesday, and it is not March and it is not Tuesday.

   (3) $(p \wedge \sim p) \wedge q$

   b. Construct two contradictions, one using symbols and the other using words.

3. One way to construct a tautology is to use a contradiction as the hypothesis of a conditional. For instance, since we know that $p \wedge \sim p$ is a contradiction, the conditional $(p \wedge \sim p) \rightarrow q$ is a tautology for any conclusion q.

   a. Construct two conditionals using any two contradictions as the premises.

   b. Show that the two conditionals from part **a** are tautologies.

   c. Explain why the method of using a contradiction as the hypothesis of a conditional always results in a tautology.

CUMULATIVE REVIEW CHAPTERS 1–2

Part I

Answer all questions in this part. Each correct answer will receive 2 credits. No partial credit will be allowed.

1. Which of the following is an undefined term?

   (1) ray (2) angle (3) line (4) line segment

2. The statement "If x is a prime then x is odd" is false when x equals

   (1) 1 (2) 2 (3) 3 (4) 4

3. Points J, K, and L lie on a line. The coordinate of J is -17, the coordinate of K is -8, and the coordinate of L is 13. What is the coordinate of M, the midpoint of $\overline{JL}$?

(1) -8 (2) -4 (3) -2 (4) 2

4. When "Today is Saturday" is false, which of the following statements could be either true or false?

(1) If today is Saturday, then I do not have to go to school.
(2) Today is Saturday and I do not have to go to school.
(3) Today is Saturday or I have to go to school.
(4) Today is not Saturday.

5. Which of the following is *not* a requirement in order for point H to be between points G and I?

(1) $GH = HI$
(2) G, H, and I are collinear.
(3) $GH + HI = GI$
(4) G, H, and I are distinct points.

6. $\overrightarrow{UW}$ bisects $\angle TUV$. If $m\angle TUV = 34x$ and $m\angle VUW = 5x + 30$, what is $m\angle TUV$?

(1) 2.5° (2) 32.5° (3) 42.5° (4) 85°

7. Which of the following must be true when $AB + BC = AC$?

(1) B is the midpoint of $\overline{AC}$
(2) $AB = BC$
(3) C is a point on $\overline{AB}$
(4) B is a point on $\overline{AC}$

8. Which of the following equalities is an example of the use of the commutative property?

(1) $3(2 + x) = 6 + 3x$
(2) $3 + (2 + x) = (3 + 2) + x$
(3) $3 + (0 + x) = 3 + x$
(4) $3(2 + x) = 3(x + 2)$

9. Which of the following must be true when p is true?

(1) $p \wedge q$ (2) $p \rightarrow q$ (3) $p \vee q$ (4) $\sim p \vee q$

10. The solution set of $x + 1 = x$ is

(1) $\varnothing$ (2) $\{0\}$ (3) $\{-1\}$ (4) {all real numbers}

Part II

Answer all questions in this part. Each correct answer will receive 2 credits. Clearly indicate the necessary steps, including appropriate formula substitutions, diagrams, graphs, charts, etc. For all questions in this part, a correct numerical answer with no work shown will receive only 1 credit.

11. The first two sentences below are true. Determine whether the third sentence is true, is false, or cannot be found to be true or false. Justify your answer.

I win the ring toss game.
If I win the ring toss game, then I get a goldfish.
I get a goldfish.

12. On the number line, the coordinate of R is -5 and the coordinate of S is -1. What is the coordinate of T if S is the midpoint of $\overline{RT}$?

Part III

Answer all questions in this part. Each correct answer will receive 4 credits. Clearly indicate the necessary steps, including appropriate formula substitutions, diagrams, graphs, charts, etc. For all questions in this part, a correct numerical answer with no work shown will receive only 1 credit.

13. Let A and B be two points in a plane. Explain the meanings of the symbols $\overleftrightarrow{AB}$, $\overrightarrow{AB}$, $\overline{AB}$, and AB.

14. The ray $\overrightarrow{BD}$ is the bisector of $\angle ABC$, a straight angle. Explain why $\overrightarrow{BD}$ is perpendicular to $\overleftrightarrow{ABC}$. Use definitions to justify your explanation.

Part IV

Answer all questions in this part. Each correct answer will receive 6 credits. Clearly indicate the necessary steps, including appropriate formula substitutions, diagrams, graphs, charts, etc. For all questions in this part, a correct numerical answer with no work shown will receive only 1 credit.

15. The steps used to simplify an algebraic expression are shown below. Name the property that justifies each of the steps.

$$\begin{aligned} a(2b + 1) &= a(2b) + a(1) \\ &= (a \cdot 2) \cdot b + a(1) \\ &= (2 \cdot a) \cdot b + a(1) \\ &= 2ab + a \end{aligned}$$

16. The statement "If x is divisible by 12, then x is divisible by 4" is always true.

a. Write the converse of the statement.

b. Write the inverse of the statement.

c. Are the converse and inverse true for all positive integers x? Justify your answer.

d. Write another statement using "x is divisible by 12" and "x is divisible by 4" or the negations of these statements that is true for all positive integers x.